# Dynamical analysis of random, quantum interference of light

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The well-known photon correlations of thermal light are now understood to result from the random superposition of independently emitted photons from spontaneous emission. Through random interference the number of photons evolves as a Bose-Einstein distribution rather than the Poisson distribution as one might expect for independent emissions of classical particles. By identifying terms in the density-matrix ($\rho$) equation for a linear amplifier, those terms giving rise to spontaneous emission were earlier distinguished from those causing amplification and absorption. We investigate the role of interference in the evolution of the photon statistics by further identifying terms in the equation for $\dot{\rho}_n$ solely responsible for quantum interference phenomena. The effects of this random interference on the photon factorial moments are quantified, even for those cases where the final field statistics are not Bose-Einstein. From our analysis we conclude that stimulated emission should be viewed as analogous to time-reversed absorption, rather than either a cascade process or pure constructive interference.

## I. INTRODUCTION

The origin of correlations in thermal light has sometimes been ascribed to stimulated emission[1] and at other times to the random interference of independently emitted light waves from spontaneous decay of excited atoms.[2] Although stimulated emission has been explained as arising from constructive interference of the emitted radiation with an incident field,[3] the former view has seemed more consistent with an interpretation of cascading photon particles, while the latter seems to require that one think of photons as waves. By identifying terms corresponding to either spontaneous emission or to linear amplification in the equation for the density matrix $\rho$ of a linear amplifier,[4] and showing how the field evolves according to each set of terms separately,[5] it has been shown that, indeed, the evolution of the quantum statistics of amplified spontaneous emission (ASE) in a linear laser amplifier (and thermal light) arises solely from independent emissions and subsequent random superposition of light. In those cases, such as a (nonlinear) laser oscillator or for a single isolated atom, where stimulated emission (and amplification) do change the photon statistics, it does not lead to Bose-Einstein statistics.

Given that Bose-Einstein statistics, responsible for the observed photon clumping of thermal light such as in the Hanbury Brown-Twiss effect,[6] arises from truly independent emissions from independent excited atoms, with no help from stimulated emission, one would expect that the number of atoms which have decayed in a macroscopic sample would be Poisson. The evolution to Bose-Einstein statistics would appear to be subsequent to and independent of the emission process.

This can also be seen by comparison with the generation of pseudothermal, Gaussian light by scattering of a laser beam from a rotating ground glass. The number of photons feeding into the scattering volume will be approximately Poisson for a laser beam far above threshold. The change in statistics for this case arises solely from the random superposition of the different parts of the beam with itself. In both systems the source is Poisson and the statistics within the mixing volume is Bose-Einstein. With a slightly different geometry one can in fact envisage a closer analogy to spontaneous emission. Consider an initially coherent beam enclosed within a cavity with a random distribution of scatterers that slowly mix or randomize the beam. This will cause the original Poisson number of photons to evolve into a Bose-Einstein distribution through constructive and destructive interference.

It is of interest to understand this process better, independent of the process that originally generated the photons. In this paper we study this evolution of the photon-number distribution by identifying those terms in the density-matrix equation which represent the effects of constructive and destructive interference in the number representation. Having such an identification for spontaneous emission in a linear amplifier, we argue that those same terms must represent the evolution of $\rho_n$ from random interference of light beams in more general situations such as the ground glass scattering of laser light, the random superposition of laser light scattered from the turbulent atmosphere, or even in situations in which there is also nonlinear amplification of the beam. The last two situations are well known to lead to non-Gaussian fields. A separate analysis of the change in the probability generating function for random superposition of photons confirms our identification.

In Sec. II we review the background assumptions, definitions, and equations. In Sec. III we present the identification of the interference terms in the Kolmogorov equation for evolution of the diagonal elements of the density matrix. Using methods developed in a companion paper[7] we show that this identification is generally valid for random superposition of light. We then present a heuristic analysis of these random quantum interference terms. Our analysis also leads to a better understanding of the meaning of the stimulated emission terms.

## II. PHYSICAL AND MATHEMATICAL BACKGROUND

Although we argue that the interference evolution should be independent of the origin of the light, we first consider a single mode of the electromagnetic field in interaction with atoms possessing a significant electric-dipole matrix element for transitions between two energy levels. The energy separation is assumed equal to the energy of one photon of the mode. For our purposes it will be sufficient to consider the atoms as having just these two levels of interest, with raising and lowering operators $\sigma^+$ and $\sigma^-$, respectively. Making the rotating-wave approximation, the interaction Hamiltonian is

$$H=\hbar g(a\sigma^\dagger+a^\dagger\sigma)\ . \qquad (1)$$

The coupling constant $g$ contains both the electric-dipole matrix element and averaged dependence on the atomic positions.

Because we will be working mainly in the photon-number representation, we first define the diagonal elements of the density matrix $\rho$ for the electromagnetic field,

$$\rho_n=\langle n\,|\rho|\,n\rangle\ , \qquad (2)$$

with probability generating function[8] (PGF),

$$G(\gamma,t)=\mathrm{tr}[\rho(1-\gamma)^{a^\dagger a}]\equiv\langle(1-\gamma)^n\rangle=\sum_{n=0}^{\infty}\rho_n(1-\gamma)^n\ . \qquad (3)$$

Convergence is guaranteed for $|1-\gamma|\leq 1$. The factorial moments,

$$n^{(k)}\equiv\langle n(n-1)\cdots(n-k+1)\rangle\ , \qquad (4)$$

are derived by repeated differentiation with respect to $-\gamma$, and setting $\gamma=0$. They are related to the ordinary moments of the field intensity[9] when they exist. The factorial moments therefore contain information about the field-intensity fluctuations. Using a Taylor series expansion of $G(\gamma,t)$ about $\gamma=0$ we can also write

$$G(\gamma,t)=\sum_{k=0}^{\infty}\frac{(-\gamma)^k n^{(k)}}{k!}\ . \qquad (5)$$

The probability generating function can also be expressed in terms of the normally ordered quantum characteristic function,

$$C(\xi,\xi^*)=\langle e^{\xi a^\dagger}e^{-\xi^* a}\rangle\ , \qquad (6)$$

by the formula[10]

$$G(\gamma)=\frac{1}{\pi\gamma}\int\int\exp\left[\frac{-|\xi|^2}{\gamma}\right]C(\xi,\xi^*)d^2\xi\ . \qquad (7)$$

In a companion paper[7] we show that the random superposition of a single-photon eigenstate with a field having a nonsingular $P$ representation (and PGF$=G_0$) can be derived from Eq. (7), yielding

$$G(\gamma)=\left[1-\gamma\frac{d}{d\gamma}\gamma\right]G_0(\gamma) \qquad (8a)$$

or

$$G(\gamma)=(1-\gamma)G_0(\gamma)-\gamma^2\frac{d}{d\gamma}G_0(\gamma)\ , \qquad (8b)$$

where $G_0$ is the PGF of the original field. The first term in Eq. (8b) obviously represents the addition of a classical particle; hence the second term describes the change in $G_0$ caused by random quantum interference. Iteration of Eq. (8a) $m$ times, describing the independent random superposition of a random number $m$ of single-photon eigenstates, leads to the equation

$$G(\gamma)=G_m\left[\gamma\frac{d}{d\gamma}\gamma\right]G_0(\gamma) \qquad (9)$$

after averaging over $m$, where $G_m$ is the PGF for the distribution of $m$.

The evolution of $\rho_n$ for a many-atom case in which the populations of the upper and lower atomic states are held constant by a pumping process can be written as[4,5]

$$\dot\rho_n=(A-B)[n\rho_n-(n+1)\rho_{n+1}]+A[(n+1)\rho_{n+1}-(2n+1)\rho_n+n\rho_{n-1}]\ . \qquad (10)$$

This corresponds to a linear laser amplifier which is understood to coherently amplify any input field, along with the addition of amplified spontaneous emission (ASE) noise.

In this equation the terms proportional to $(A-B)$ and $A$ describe amplification or absorption and noise generation through spontaneous emission, respectively. Because $A$ and $B$ are independent, this regrouping uniquely determines a corresponding regrouping of the operator equation for $\rho$. This is different from the usual identification of stimulated emission in which those coefficients containing $A$ are modified by changing $n+1$ to $n$, thus presumably separating out the effect of spontaneous emission. As was shown, however, such an identification yields a Kolmogorov[11] (or master) equation for spontaneous emission (the subtracted terms),

$$\dot\rho_n=A[-\rho_n+\rho_{n-1}]\ , \qquad (11)$$

which leads to Poisson statistics (i.e., independent particle behavior) for the spontaneous emissions. This seems to imply (incorrectly) that it must be stimulated emission that leads to the clumping of photons measured in the Hanbury Brown-Twiss effect and characteristic of Bose-Einstein statistics. The problem is in the neglect of the random interference of the independently emitted photons. The second grouping of terms in Eq. (10), however, does include the interference effects. In fact, when $A=B$ so that there is no net amplification or decay of the field, the remaining spontaneous emission terms cause an initial vacuum field to evolve into one with Bose-Einstein statistics.

It was also shown in Ref. 5 that the evolution of the field under the terms corresponding to stimulated emission and absorption (the $A-B$ term) could be described by a PGF of the form

$$G(\gamma,t)=G_0[\gamma e^{\delta t}] , \tag{12}$$

where $\delta \equiv A - B$. It is easily verified that this preserves the normalized factorial moments,

$$\frac{\langle n(n-1)(n-2)\cdots(n-m+1)\rangle}{\langle n\rangle^m} ,$$

hence, the nature of the photon statistics is not changed by either absorption or linear amplification.

One argument for the earlier identification of stimulated emission terms relied on the picture of incident photon particles stimulating the emission of secondary photons by excited atoms. It is easy to see how this cascade process seemed to imply a clumping of the photons in the resulting field, hence a positive correlation among them. However, as we have seen, our identification of amplification and absorption terms was based on an analysis showing that the clumping can be understood as arising solely from the random interference of the spontaneous emissions. Hence, the laser amplifier, in fact, does amplify coherently, and the $(A-B)$ terms contribute no additional correlations in the photons.

In spite of this, it has been difficult to understand conceptually how, in the case when the amplification and absorption just balance out (so that $A=B$), there are still no residual effects on the photon statistics from generation-recombination noise. In the general model of generation-recombination in which particles are created or destroyed (one at a time) the Kolmogorov equation can be written[12]

$$\dot{\rho}_n = r(n+1)\rho_{n+1} - [g(n)+r(n)]\rho_n + g(n-1)\rho_{n-1} . \tag{13}$$

It is easy to see that setting $g(n)=r(n)$, although maintaining a constant average number of particles $\bar{n}(t)$, leaves $\dot{\rho}_n \neq 0$. Hence, the photon statistics will still change with time. This is more readily seen by the following analysis. For large values of $\bar{n}$ and slowly varying generation and recombination rates, $g(n)$ and $r(n)$, one can make the diffusion approximation for the unit shift operator,

$$\exp(\pm\partial/\partial n) \approx 1 \pm \partial/\partial n + \tfrac{1}{2}(\partial/\partial n)^2 + \cdots , \tag{14}$$

to obtain the corresponding Fokker-Planck equation (FPE),

$$\dot{\rho}(n,t) = -\frac{\partial}{\partial n}[\{g(n)-r(n)\}\rho(n,t)] + \frac{1}{2}\frac{\partial^2}{\partial n^2}[\{g(n)+r(n)\}\rho(n,t)] , \tag{15}$$

describing diffusion in $n$ space. This is in the Ito form, hence the corresponding stochastic differential equation is[12]

$$\dot{n} = \{g(n)-r(n)\}n + \sqrt{\{g(n)+r(n)\}}\,\eta(t) , \tag{16}$$

where $\eta$ is white Gaussian noise. From this one more readily sees the stochastic evolution of $n$, even when $g(n)=r(n)$.

In a linear laser amplifier absorption and re-emission would seem to continue even when their average effects cancel, i.e., $A=B$. Hence, from the above, one might still expect some observable effect from this noise in the evolution of the photon statistics.

## III. RANDOM FIELD INTERFERENCE TERMS

### A. Identification of interference evolution

We now analyze the spontaneous emission terms in more detail. Because the second terms in Eq. (10) account for both the independent emission of photons by the atoms and the subsequent interference of those photons, it seems reasonable to further separate them into

$$A[(n+1)\rho_{n+1} - 2n\rho_n + (n-1)\rho_{n-1}] + A[-\rho_n + \rho_{n-1}] .$$

Here the terms in the second brackets may be recognized as those giving rise to a Poisson process and are thus identified with the independent emission of photons. That leaves the remaining terms in the first brackets to be identified with the evolution of photon-number statistics resulting from random constructive and destructive interference. It is easy to see that the latter terms conserve the average number of photons. This can be depicted as in Fig. 1, in which the transition rates between levels are given by $A$ times the numbers written next to each arrow. Note that we have again identified pure, independent spontaneous emissions with the Poisson terms in the Kolmogorov equation. According to this description, the spontaneous emissions truly originate as independent emissions and only later interfere randomly to yield Bose-Einstein statistics. This is what one would expect from widely separated atoms prepared independently in the upper state. One could imagine that the independently emitted fields could be gathered and channeled into a given mixing volume where they eventually overlap and randomly interfere. The implication is clear that the number of atoms having decayed has a Poisson distribution, and this is not changed by subsequent interference of the electromagnetic field.

Viewing the random interference process as a generation-recombination process, the corresponding master (or Kolmogorov) equation for both constructive and destructive interference is immediately written down, for a general process, where $\Delta m$ photons are randomly

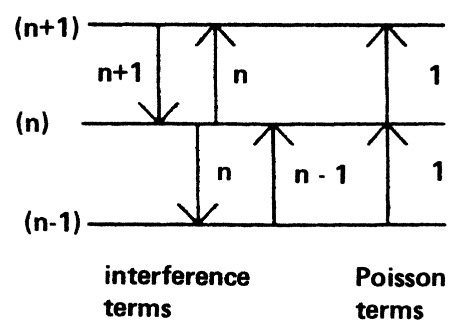


FIG. 1. Transition rates for spontaneous emission.

mixed in a time $\Delta t$, as

$$\dot{\rho}_n = \frac{\Delta m}{\Delta t}[(n+1)\rho_{n+1} - 2n\rho_n + (n-1)\rho_{n-1}] , \quad (17)$$

or, in a more general form, as

$$\frac{d\rho_n}{dm} = [(n+1)\rho_{n+1} - 2n\rho_n + (n-1)\rho_{n-1}] . \quad (18)$$

This represents the evolution of the diagonal elements of the density matrix under chaotic mixing of the original field.

Using standard methods,[11,12] the corresponding equation for the PGF is

$$\dot{G}(\gamma,t) = -\lambda\gamma^2 \frac{d}{d\gamma} G(\gamma,t) , \quad (19)$$

where $\lambda \equiv \Delta m / \Delta t$. The solution of this equation is

$$G(\gamma,t) = G_0\left[\frac{\gamma}{1+\gamma\lambda t}\right] . \quad (20)$$

Clearly the interference-related evolution cannot go on indefinitely as it is not independent of the process that actually causes the mixing, such as the independent spontaneous emissions of atoms or the random superposition caused by laser scattering in the turbulent atmosphere. Note the similarity of Eq. (20) to the equation giving the addition of thermal noise, with mean photon number $\bar{n}_T$, to an arbitrary initial field having PGF $G_0(\gamma)$,[13]

$$G(\gamma) = \frac{1}{1+\bar{n}_T\gamma} G_0\left[\frac{\gamma}{1+\bar{n}_T\gamma}\right] . \quad (21)$$

Let the mixing in Eq. (20) continue until an average number, $\lambda t = \bar{n}_T$, of photons have been mixed, and then add a Bose-Einstein distribution of classical particles (i.e., without interference) having PGF $1/(1+\bar{n}_T\gamma)$ by simply multiplying the PGF's. We recover Eq. (21) exactly, having thus separated the interference-induced change in $G$ from that caused by addition of classical particles.

We can also obtain the identification of interference summarized in Eq. (19) in a more general context using Eq. (8a). Assume that the addition of photons obeys a Poisson process with rate $\lambda$. Hence, the probability of a photon being added to the existing field in time $\Delta t$ is $\lambda\,\Delta t$ (to order $\Delta t$). The PGF is then easily shown to obey

$$G(\gamma,t+\Delta t) = (1-\lambda\,\Delta t)G(\gamma,t) + \lambda\,\Delta t\left[(1-\gamma) - \gamma^2\frac{d}{d\gamma}\right]G(\gamma,t) , \quad (22)$$

or, in the limit as $\Delta t \to 0$,

$$\dot{G}(\gamma,t) = -\lambda\gamma G(\gamma,t) - \lambda\gamma^2\frac{d}{d\gamma}G(\gamma,t) . \quad (23)$$

Hence, the first term arises from the classical addition of particles (the Poisson part) and the second, as discussed in Sec. II, describes the effect of random, quantum interference.

We can now make some quite general statements concerning the effects of quantum interference, independent of the details of the source of the photons randomly superposed. Assume the original field is the vacuum in Eq. (9) for the general superposition of a random number $m$ of photons, i.e., $G_0 = 1$. Then we obtain

$$G(\gamma) = G_m\left[\gamma\frac{d}{d\gamma}\gamma\right]1 , \quad (24)$$

which can be expressed in terms of the factorial moments of $m$, denoted by $n_{\rm cl}^{(k)}$ (cl denotes classical, i.e., $m \equiv n_{\rm cl}$), with the help of Eq. (5) to expand $G_m$,

$$G(\gamma,t) = \sum_{k=0}^{\infty} \frac{\left[-\gamma\frac{d}{d\gamma}\gamma\right]^k n_{\rm cl}^{(k)}}{k!} . \quad (25)$$

Now, it can be verified using induction that

$$\left[\gamma\frac{d}{d\gamma}\gamma\right]^k 1 = k!\gamma^k ; \quad (26)$$

hence, comparing with Eq. (5) again, but now for the expansion of $G$ in terms of its factorial moments, $n_Q^{(k)}$ ($Q$ denotes quantum), we find

$$n_Q^{(k)} = k!\,n_{\rm cl}^{(k)} . \quad (27)$$

In particular, the averages are always equal, i.e., $\bar{n}_Q = \bar{n}_{\rm cl}$. For example, if $m$ has a Poisson distribution (from spontaneous emission or laser scattering from a rotating ground glass), then

$$n_{\rm cl}^{(k)} = (\bar{m})^k ; \quad (28)$$

hence, from Eq. (27),

$$n_Q^{(k)} = k!(\bar{m})^k , \quad (29)$$

which are the factorial moments of a Bose-Einstein (geometric) distribution.

Another example, of interest in the study of fluctuations in laser light propagating through the atmosphere, is when the number of photons has a negative binomial distribution appropriate when the source is a random number of scattering elements in the turbulent atmosphere, evolving according to a birth-death–immigration process.[14] A semiclassical analysis of this model leads to what is termed $K$-distributed amplitude fluctuations. The factorial moments of a negative binomial distribution (parameter $\beta$) are easily obtained by observing that it can be thought of as the sum of $\beta$ geometric random variables ($\beta$ need not be an integer). Hence, the PGF is

$$G_m(\gamma) = \left[1+\frac{\bar{n}_{\rm cl}\gamma}{\beta}\right]^{-\beta} , \quad (30)$$

and differentiating *wrt* $(-\gamma)$ $k$ times we have,

$$n_{\rm cl}^{(k)} = \frac{\Gamma(k+\beta)(\bar{n}_{\rm cl})^k}{\Gamma(\beta)\beta^k} . \quad (31)$$

Hence, the factorial moments of the field arising from a random superposition of a negative binomial number of photons is

$$n_Q^{(k)} = k! \frac{\Gamma(k+\beta)(\bar{n}_{cl})^k}{\Gamma(\beta)\beta^k} , \tag{32}$$

which agrees with intensity fluctuations found for the $K$ distribution.[15]

Considering the second factorial moment from Eq. (27), we see that the quantum fluctuations of the number of photons arising from random superposition is always twice that of the underlying classical distribution. Hence, the Hanbury Brown-Twiss effect is not just qualitatively present but, if properly defined, is quantitatively the same, for arbitrary, non-Gaussian, noise fields. From the derivation it is clear that during the superposition the photon statistics evolve to those expressed in Eq. (27) according to the interference terms summarized in Eq. (18).

### B. Heuristic interpretation

To better understand the identification of the random interference terms we suggest the following heuristic model of the interference process. Imagine a gradual mixing of the electromagnetic field within a given volume. Let $\Delta m$ represent the number of photons corresponding to the amount of the field randomly mixed in time $\Delta t$. Now, in the usual heuristic analysis of superposition of two wave packets (two photons) the interference is sometimes considered to be either completely destructive, resulting in complete cancellation (zero photons) or completely constructive, resulting in a doubling of the amplitude and hence four photons. However, in chaotic mixing it is relatively unlikely that either extreme will be realized, but rather that the interference will either be slightly constructive or slightly destructive. Hence, of the five possible final number of photons (zero, one, two, three, four) resulting from random interference of two photons, the most likely number is still two, with some probability of a net increase or decrease of one photon and negligible probability of a change of $\pm 2$ photons. If the field consists of $n$ photons and the rate of mixing corresponds to $\Delta m/\Delta t$ photons per unit time, then it is reasonable to suppose that the transition rate either up or down by one photon will be proportional to $n\,\Delta m/\Delta t$ (neglecting second-order differentials in the numerator). Hence, the transition rate diagram for the purely destructive part of this random interference would look like that in Fig. 2, while the diagram for purely constructive interference would be as presented in Fig. 3. Here the transition rates are $\Delta m/\Delta t$ times the numbers next to each arrow. For destructive interference the corresponding Kolmogorov equation is of the same form as the absorption terms in Eq. (10),

$$\dot{\rho}_n = -\lambda[n\rho_n - (n+1)\rho_{n+1}] , \tag{33}$$

where $\lambda = \Delta m/\Delta t$.

We saw that the probability generating function for this equation (Eq. 12 with $\delta \to -\lambda$) conserves the value of the normalized factorial moments; hence, the nature of the photon statistics of an arbitrary initial field are not changed by either linear absorption or, equivalently, pure destructive interference. Viewing this process as a generation-recombination process (with no generation),

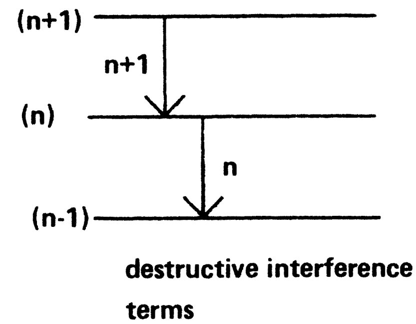


FIG. 2. Transition rates for random destructive interference.

one might expect there to be some recombination noise. In fact, again making the diffusion approximation to obtain the corresponding Fokker-Planck equation, we obtain

$$\dot{\rho}(n,t) = \frac{\partial}{\partial n}[\lambda n \rho(n,t)] + \frac{1}{2}\frac{\partial^2}{\partial n^2}[\lambda n \rho(n,t)] , \tag{34}$$

with the corresponding stochastic differential equation

$$\dot{n} = -\lambda n + \sqrt{\lambda n}\,\eta(t) . \tag{35}$$

From this one sees that the diffusion does remain, but evidently the diffusion parameter is of precisely the correct magnitude to insure the constancy of the normalized moments. However, the equation corresponding to Eq. (33) in the $P$ representation (to obtain which, no diffusion approximation is necessary),

$$\dot{P}(\alpha,t) = \frac{\lambda}{2}\left[\frac{\partial}{\partial\alpha}\alpha P(\alpha,t) + \frac{\partial}{\partial\alpha^*}\alpha^* P(\alpha,t)\right] , \tag{36}$$

is not a FPE inasmuch as it contains no diffusion term, thus more directly showing the constant nature of the photon statistics. The corresponding differential equations

$$\dot{\alpha} = -\frac{\lambda}{2}\alpha \tag{37a}$$

and

$$\dot{\alpha}^* = -\frac{\lambda}{2}\alpha^* \tag{37b}$$

are, in fact, deterministic. Hence, we see that for pure

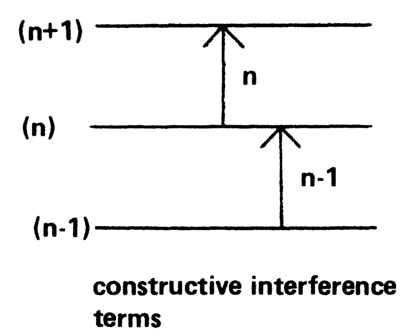


FIG. 3. Transition rates for random constructive interference.

destructive interference (or liner absorption) the field-amplitude decay is also deterministic.

Turning now to the pure constructive interference process we recognize it as simply the Yule-Furry or linear growth process[16] with the Kolmogorov equation

$$\dot{\rho}_n = \lambda[-n\rho_n + (n-1)\rho_{n-1}] \ . \tag{38}$$

Using standard methods the probability generating function is found to be

$$G(\gamma,t) = G_0\left[\frac{\gamma e^{-\lambda t}}{1+\gamma(e^{-\lambda t}-1)}\right] \tag{39}$$

for an arbitrary initial distribution. To find the PGF starting from exactly one photon, we use

$$G_0(\gamma) = (1-\gamma)^1 \ , \tag{40}$$

hence,

$$G(\gamma,t) = \frac{1-\gamma}{1+\gamma(e^{\lambda t}-1)} \ . \tag{41}$$

The PGF of daughters is obtained by multiplying by $(1-\gamma)^{-1}$ which effectively subtracts 1 (the original photon) from $n$, yielding

$$G(\gamma,t) = \frac{1}{1+\gamma(e^{\lambda t}-1)} \ . \tag{42}$$

This is the PGF of a geometric or Bose-Einstein distribution with mean

$$\bar{n}(t) = (e^{\lambda t}-1) \ . \tag{43}$$

It is straightforward to show, however, that a Bose-Einstein distribution is not preserved by this process. In fact, if the number of original photons is definitely $n_0$, then the number of daughters is the sum of $n_0$ geometric random variables, i.e., it has a negative binomial distribution.

We see that while pure destructive interference preserves the nature of the photon statistics, pure constructive interference changes the statistics and enhances photon clumping, in fact, as most clearly seen for at least one case, leading to Bose-Einstein types of correlations.

We have noted that the destructive interference terms are the same as those corresponding to absorption of the laser field in Eq. (10). The net effect of destructive interference is identical to that of absorption of protons, e.g., by the lower state of the lasing transition, or generally by any linear absorption. When $A > B$ in Eq. (10) the terms (now corresponding to amplification) no longer constitute a valid Kolmogorov equation by themselves.[11] Hence, they do not really constitute the logical "opposite" of absorption (or equivalently the opposite of pure destructive interference) which we showed to be constructive interference (the Yule-Furry process) as one might have supposed. Instead, they are now proportional to the negative of the destructive interference and tend to cancel those terms in the "interference" part of the equation. In fact, if one takes $B$ equal to zero (no absorption), Eq. (10) can be written

$$\begin{aligned}\dot{\rho}_n = {} & A[n\rho_n-(n+1)\rho_{n+1}] \ \text{(stimulated emission)}\\ & + A[(n+1)\rho_{n+1}-2n\rho_n\\ & \quad +(n-1)\rho_{n-1}] \ \text{(interference)}\\ & + A[-\rho_n+\rho_{n-1}] \ \text{("spontaneous" emission)} \ .\end{aligned} \tag{44}$$

From this one readily sees that stimulated emission has the form of time reversed absorption, rather than constructive interference.

Canceling terms, we have

$$\dot{\rho}_n = A[-n\rho_n+(n-1)\rho_{n-1}] + A[-\rho_n+\rho_{n-1}] \ . \tag{45}$$

The form of this equation is misleading, although it is, in fact just the sum of pure constructive interference and the Poisson (spontaneous emission) terms. It is what one would write down directly for stimulated and spontaneous emission viewed as a particle cascade process (Yule-Furry or pure constructive interference) plus independent emission of photons, respectively. From the previous analysis we interpret this as the fact that two errors of interpretation (1) neglecting interference of the spontaneous emission, and (2) incorrecty identifying stimulated emission as a classical cascade process, can cancel, thus leading to the correct result. We argue that this fact has contributed to the confusion concerning the nature of interference and stimulated emission in the past.

It is partly because the processes of spontaneous emission, interference, and stimulated emission are inseparable in a fully quantized theory that these processes are so difficult to understand. To better understand stimulated emission as identified in Eq. (10), imagine running a movie of the absorption process backwards. We should not be surprised to then see a process of coherent amplification. However, the time reversal of a valid Kolmogorov equation cannot be another valid Kolmogorov equation since the sign of $\lambda_{nn}$ is reversed.[11] By contrast, the Yule-Furry process of "amplification" *is* a valid random Markov process. Hence it cannot be the time reversal of the absorption process. From these arguments one also sees that stimulated emission cannot exist alone. It must appear in the presence of spontaneous emission to preserve the unitarity of the theory.

## ACKNOWLEDGMENT

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[3]For example, see W. E. Lamb, Jr., in *Quantum Optics and Electronics,* Proceedings of the Les Houches Summer School of Physics, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1964).

[4]N. B. Abraham and S. R. Smith, Phys. Rev. A **15**, 421 (1977); note that their Eq. (34) for stimulated emission or absorption, to remain normalized, should be

$$P(\alpha,t)=P(\alpha e^{-\gamma t/2},0)e^{-\gamma t} .$$

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[6]R. Hanbury Brown and R. Q. Twiss, Nature **177**, 27 (1956); Proc. R. Soc. Lond. **242**, 300 (1957); **243**, 291 (1957).

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[8]An alternative definition of the probability generating function replaces $1-\gamma$ with $z$. There is no essential difference between the two definitions.

[9]J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), pp. 21–26.

[10]E. B. Rockower and N. B. Abraham, J. Phys. A **11**, 1879 (1978); see also Ref. 7.

[11]A Kolmogorov equation is a first-order time evolution equation of the form

$$\frac{d}{dt}P_n=\sum_m \lambda_{mn}P_m ,$$

where $\sum_n \lambda_{mn}=0$ and $\lambda_{nn}<0$. Such an equation governs a Markov process [see A. B. Clark and R. L. Disney, *Probability and Random Processes for Engineers and Scientists* (Wiley, New York, 1970), p. 281; W. Feller, *An Introduction to Probability and its Applications,* 3rd ed. (Wiley, New York, 1968), Vol. I, Chap. XVII]. These are exactly the conditions for the time evolution of a density matrix for which the first condition conserves probability while the second guarantees $0\le\rho_n\le 1$ for all time. This is often called a master equation for a discrete-state continuous-time Markov process [see N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, New York, 1981), p. 102].

[12]See, for instance, C. W. Gardiner, *Handbook of Stochastic Methods,* 2nd ed. (Springer-Verlag, New York, 1985).

[13]See the companion paper, Ref. 7.

[14]E. Jakeman and P. N. Pusey, Phys. Rev. Lett. **40**, 546 (1978); G. Parry and P. N. Pusey, J. Opt. Soc. Am. **69**, 796 (1979); E. Jakeman, J. Phys. A **13**, 31 (1980).

[15]See the article by Jakeman in Ref. 14. This is analyzed further in E. B. Rockower, J. Opt. Soc. Am. A (to be published).

[16]D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes* (Wiley, New York, 1965), p. 156ff.