Calculating the quantum characteristic function and the photon-number generating function in quantum optics

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A new operator derivation of the relation giving the photon-number generating function, $G(\gamma)$, in terms of the quantum characteristic function, $C(\xi,\xi^*)$, is presented. The inverse problem is then solved, calculating $C(\xi,\xi^*)$ directly from $G(\gamma)$. Because $G(\gamma)$ contains less phase information than $C(\xi,\xi^*)$, we can either assume that the field has a completely random phase (e.g., a stationary field) or be content with calculating the phase average of $C(\xi,\xi^*)$. We then derive an expression giving $G(\gamma)$ for the superposition of two arbitrary, independent fields in terms of the individual $G(\gamma)$'s for each (stationary) field. A number of examples illustrate the methods, including a determination of the quantum characteristic function for a field with K-distributed amplitude fluctuations.

I. INTRODUCTION AND BACKGROUND

In an earlier report[1] a method was presented for calculating the number generating function directly from the quantum characteristic function for the electromagnetic field.[2] It was argued that such a relation was useful in that it allowed calculation of photocounting statistics directly from the quantum characteristic function. Moreover, it permitted direct comparison of theories yielding a characteristic function with others yielding only the generating function. For the superposition of independent fields the characteristic function is often assumed to be just the product of the individual field characteristic functions (not always justified, see Appendix B) and hence is often more easily obtained. On the other hand, the photon-number generating function for the density matrix ρ, defined by the trace,

$$G(\gamma)=\mathrm{tr}[\rho(1-\gamma)^{a^\dagger a}]=\sum_{n=0}^{\infty}\rho_n(1-\gamma)^n , \tag{1}$$

contains less, but more directly usable, information than the normally ordered quantum characteristic function. The latter is defined by[3]

$$C(\xi,\xi^*)=\mathrm{tr}(\rho e^{\xi a^\dagger}e^{-\xi^* a}) . \tag{2}$$

In these equations $a^\dagger$ and a are the creation and annihilation operators, respectively, for a single mode of the electromagnetic field. Convergence of the sum in Eq. (1) is guaranteed for $|1-\gamma|\le 1$. In the following $G(\gamma)$ will be understood to represent both the sum in Eq. (1) and its analytic continuation.

In the original derivation it was also argued that such a relation would aid in the interpretation of complicated generating functions where the contributions of independent sources (so transparent in the product form of the characteristic function) were obscured. Such an application was in fact made there concerning identification of the effects of noise in a photodetector illuminated by a multimode field. The main result of that paper was the relation

$$G(\gamma)=\frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-|\xi|^2}{\gamma}\right]C(\xi,\xi^*)d^2\xi . \tag{3}$$

In our original derivation of Eq. (3) we required the existence of all factorial moments. Perina[4] then contributed a proof valid in a space of generalized functions when antinormal ordering is adopted (a quasidistribution relative to coherent states). By contrast, the derivation presented in Sec. II obtains the relation from purely operator manipulations. The new derivation is not only more general, but invites generalizations producing additional identities.

In Sec. III we show that it is possible to invert Eq. (3) if one is willing to make an assumption about the distribution of phase in the field. Hence, given the photon-number generating function and (for instance) assuming random phase for the field (valid for a stationary field), we show how to calculate the corresponding quantum characteristic function. With this result, just a Fourier transformation is needed to obtain $P(\alpha)$, assuming the field has a valid P representation. We also present a formula for the change in $G(\gamma)$ caused by the superposition of a field with exactly one photon. In Sec. IV we illustrate the method of inversion with a number of examples and then apply it to determining the quantum characteristic function for fields having K-distributed amplitude fluctuations.

It is well known that the generating function for the sum of two groups of independent classical particles is just the product of their individual generating functions. Because of interference effects, the situation is more complicated for quantum particles such as photons. In Sec. V we show how to generalize the classical result to one for quantum particles by combining Eq. (3) with its inverse from Sec. III. We obtain an expression for the generating function for the superposition of arbitrary independent stationary fields in terms of the individual generating functions for each field. This expression is valid under the same conditions as the validity of multiplying characteristic functions of independent fields to obtain the characteristic function for the superposed fields. This includes all fields with a nonsingular P representation. The operational meaning of this is described.

II. CALCULATION OF THE PHOTON-NUMBER GENERATING FUNCTION FROM THE QUANTUM CHARACTERISTIC FUNCTION

A. An operator derivation of the formula

We now present a more general derivation of Eq. (3). The technique is to promote a parameter in an integral identity to an operator, taking care to account for the fact that the operators do not in general commute. Similar techniques can also be applied to obtain a variety of useful relations involving classical random variables.[5]

Consider the following integral for c-numbers:

$$\frac{1}{\pi\gamma}\int\int \exp\left[-\frac{|\xi|^2}{\gamma}+\xi\mu-\xi^*\nu\right]d^2\xi=\exp(-\gamma\mu\nu)\ . \tag{4}$$

We can substitute harmonic-oscillator creation and annihilation operators, $a^\dagger$ and a, for μ and ν, respectively, if we define the relation in terms of normal ordering. Ambiguities with respect to the ordering of a and $a^\dagger$ are thus avoided by explicitly specifying the order in which they act. Normal ordering, in which all factors of a appear to the right of all factors of $a^\dagger$ is denoted with colons, as in:$F(a,a^\dagger)$:. We thus obtain the operator equation

$$\frac{1}{\pi\gamma}\int\int \exp\left[-\frac{|\xi|^2}{\gamma}\right]e^{\xi a^\dagger}e^{-\xi^* a}d^2\xi=\,:\exp(-\gamma a^\dagger a):\,. \tag{5}$$

The right-hand side can be rewritten without normal ordering using the well-known relation[6,7]

$$:\exp(-\gamma a^\dagger a):\,=(1-\gamma)^{a^\dagger a}\ , \tag{6}$$

thus obtaining the operator equation corresponding to Eq. (3),

$$\frac{1}{\pi\gamma}\int\int \exp\left[-\frac{|\xi|^2}{\gamma}\right]e^{\xi a^\dagger}e^{-\xi^* a}d^2\xi=(1-\gamma)^{a^\dagger a}\ . \tag{7}$$

To recover Eq. (3) just multiply by the field density matrix ρ and perform the trace.

Other, more general relations can also be obtained from Eq. (7). For example, multiplying it by $e^{\omega a^\dagger}$ from the left and by $e^{-\omega^* a}$ from the right before tracing over the density matrix, and the change of variables, $\xi+\omega\to\xi$, leads to a more general type of generating function[8] in terms of the characteristic function

$$\frac{1}{\pi\gamma}\int\int \exp\left[-\frac{|\xi-\omega|^2}{\gamma}\right]C(\xi,\xi^*)d^2\xi =\langle e^{\omega a^\dagger}(1-\gamma)^{a^\dagger a}e^{-\omega^* a}\rangle\ . \tag{8}$$

B. Application of the formula

In Appendix A we derive an expression for the generating function of the field resulting from the addition of thermal noise to an arbitrary field. The expression we obtain there,

$$G(\gamma)=\frac{1}{1+\bar{n}_T\gamma}G_0\left[\frac{\gamma}{1+\bar{n}_T\gamma}\right]\ , \tag{9}$$

is a special case of one obtained[9] when the gain equals the losses in a linear laser amplifier.[10] The derivation in Appendix A is much simpler than either and, moreover, does not rely on assumptions about how the mixing of the fields is effected (as in the former reference) nor on the existence of all factorial moments (as in the latter reference).

In Appendix B we discuss conditions under which the characteristic function for the superposition of two independent fields is simply the product of the individual characteristic functions. Assuming such conditions are met, we obtain a formula for the generating function when a one-photon state is mixed with an arbitrary field, for example, during spontaneous emission by a single atom. First, we calculate the characteristic function for the one-photon state. Insert $\rho=|1\rangle\langle 1|$ in the definition, Eq. (2), of the characteristic function and obtain

$$C_1(\xi,\xi^*)=\langle 1|e^{\xi a^\dagger}e^{-\xi^* a}|1\rangle \tag{10}$$

or

$$C_1(\xi,\xi^*)=\langle 0|ae^{\xi a^\dagger}e^{-\xi^* a}a^\dagger|0\rangle\ , \tag{11}$$

in terms of creation and annihilation operators, $a^\dagger$ and a. Now use the shift property of the exponential operators[11] to write this as

$$C_1(\xi,\xi^*)=\langle 0|(a+\xi)(a^\dagger-\xi^*)|0\rangle\ , \tag{12}$$

and, finally

$$C_1(\xi,\xi^*)=1-|\xi|^2\ . \tag{13}$$

The characteristic function for the total field is then

$$C(\xi,\xi^*)=\{1-|\xi|^2\}C_0(\xi,\xi^*)\ , \tag{14}$$

where C_0 is the characteristic function of the original field. Inserting this in Eq. (3),

$$G(\gamma)=\frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-|\xi|^2}{\gamma}\right]\{1-|\xi|^2\}\times C_0(\xi,\xi^*)d^2\xi\ , \tag{15}$$

we see that it can be expressed as

$$G(\gamma)=\frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[1-\gamma^2\frac{d}{d\gamma}\right]\exp\left[\frac{-|\xi|^2}{\gamma}\right]\times C_0(\xi,\xi^*)d^2\xi\ . \tag{16}$$

Now, assuming that the interchange of order of integration and differentiation is justified by the uniform convergence of the integral, we have

$$G(\gamma)=\frac{1}{\pi\gamma}\left[1-\gamma^2\frac{d}{d\gamma}\right]\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-|\xi|^2}{\gamma}\right]C_0(\xi,\xi^*)d^2\xi\ . \tag{17}$$

This can now be expressed in terms of G_0 as

$$G(\gamma)=\left[1-\gamma\frac{d}{d\gamma}\gamma\right]G_0(\gamma)\ . \tag{18a}$$

For comparison, we present the corresponding formula for fields superposed by transmission and reflection through a partially silvered mirror with reflection and transmission coefficients δ and ϵ,

$$G(\gamma)=\left[1-|\delta|^2\gamma\frac{d}{d\gamma}\gamma\right]G_0(|\epsilon|^2\gamma)\ . \tag{18b}$$

The latter equation is valid without limitation on the original field. Also note as $\delta\to 1$ and $\epsilon\to 0$, Eq. (18b) becomes equivalent to Eq. (18a) with $G_0(\gamma)$ replaced by $G_0(|\epsilon|^2\gamma)$, in effect imposing a limitation on the (actual) field superposed with the single-photon Fock state. These matters are discussed further in Appendix B. For simplicity of the resulting equations (no factors of δ and ϵ) and because the results have more general interest for many physical situations (see explanation in Appendix B) we proceed using Eq. (18a).

If this operation of adding single photons is iterated m times, by raising the bracket to the mth power, it represents m-independent, randomly phased photons added to the original field, not the superposition of the Fock state $|m\rangle$ with the original field. Assuming m is a random variable with probability generating function $G_m(\omega)$, we obtain, by averaging over m,

$$G(\gamma)=\sum_{m=0}^{\infty}p_m\left[1-\gamma\frac{d}{d\gamma}\gamma\right]^m G_0(\gamma) \tag{19}$$

or

$$G(\gamma)=G_m\left[\gamma\frac{d}{d\gamma}\gamma\right]G_0(\gamma)\ . \tag{20}$$

We emphasize that this is not the superposition of two fields with generating functions G_m and G_0. That formula is derived in Sec. V. The equation corresponding to Eq. (20) for the P representation is obtained in Appendix B.

An example using Eq. (20) is found by choosing $G_0(\gamma)=1$, i.e., the vacuum state, and $G_m(\omega)$ as that for a Poisson distribution ($\exp[-n\omega]$) as is appropriate for independent spontaneous emissions or a laser (operating well above threshold) scattering from a rotating ground glass. By induction we find that

$$\left[\gamma\frac{d}{d\gamma}\gamma\right]^n 1=n!\gamma^n$$

and hence obtain the generating function for a geometric (Bose-Einstein) distribution $[(1+\bar{n}\gamma)^{-1}]$.

Returning to Eq. (18a) and distributing the derivative we have

$$G(\gamma)=(1-\gamma)G_0(\gamma)-\gamma^2\frac{d}{d\gamma}G_0(\gamma)\ . \tag{21}$$

The first term accounts for the addition of one classical particle, the second expresses the effect of quantum-mechanical interference when the particles are, in fact, bosons. Identifying the coefficients of powers of $(1-\gamma)$ in Eq. (21) leads to the corresponding equation for the change in ρ_n,

$$\rho_n'-\rho_n=(-\rho_n+\rho_{n-1})+[(n+1)\rho_{n+1}-2n\rho_n+(n-1)\rho_{n-1}]\ . \tag{22}$$

The first parentheses on the right-hand side correspond to the classical change in the probabilities while terms in the square brackets come from the term, $[-\gamma^2(d/d\gamma)G_0(\gamma)]$, and hence represent the effects of random, quantum interference. Assuming that the process adding photons obeys a Poisson process, rate A, the probability for adding a photon in Δt is $A\,\Delta t$. Hence, the change in ρ_n is given by the right-hand side of Eq. (22) with probability $A\,\Delta t$, and is zero with probability $(1-A\,\Delta t)$. This yields a differential equation as $\Delta t\to 0$,

$$\dot{\rho}_n=A(-\rho_n+\rho_{n-1})+A[(n+1)\rho_{n+1}-2n\rho_n+(n-1)\rho_{n-1}]\ . \tag{23}$$

This equation is identical (except for a reordering of terms) with one for spontaneous emission obtained previously;[12] moreover, we have explicitly identified the terms corresponding to interference effects. This is analyzed more fully in a companion paper.[13]

Equation (18a) must be used only for the appropriate physical superposition process as it leads to probabilities greater than one and less than zero when the original state is extremely nonclassical, such as one with a definite photon number. This can be seen quite easily from Eq. (22) by assuming $\rho_m=1$ and $\rho_n=0$ for $n\neq m$. On the other hand, no such problems arise for Eq. (23), the corresponding differential equation. The restrictions on the original state are discussed in Appendix B where it is shown that it is sufficient that either (a) the original field has a nonsingular P representation, or (b) the probability for addition of a photon in Δt may not exceed a certain value. The physical process of superposition corresponding to this is also explained there.

III. THE INVERSE PROBLEM: FIELD AMPLITUDE STATISTICS FROM PHOTON-COUNTING STATISTICS

Inasmuch as the number generating function contains no phase information about the field, the simplest assumption is that the phase is uniformly distributed in $[0,2\pi]$. Examples for which this is true are (1) any stationary field,[14] (2) the ensemble average of the laser field (because of phase diffusion from spontaneous emission), and (3) the thermal field. It is straightforward to show that this implies that $C(\xi,\xi^*)$ is also independent of the phase angle φ defined by writing the complex quantity ξ in polar form (ρ,φ). For convenience we show this in the P representation, although the results of this section do not depend on its existence. The characteristic function can be expressed in terms of $P(\alpha)$ as

$$C(\xi,\xi^*)=\int\int P(\alpha)e^{\xi\alpha^*}e^{-\xi^*\alpha}d^2\alpha \; . \tag{24}$$

Writing α in polar coordinates (r,θ) we can rewrite Eq. (24) as

$$C(\rho,\varphi)=\int_0^\infty r\,dr\int_0^{2\pi}d\theta\,P(r,\theta)e^{2ir\rho\sin(\theta-\varphi)} \; . \tag{25}$$

Now, with the assumption that $P(r,\theta)=P(r)$, i.e., it is independent of θ, we can change the variable of angular integration to $\theta'=\theta-\varphi$, and φ no longer appears on the right-hand side (inasmuch as the integration is over $0\rightarrow 2\pi$). Hence $C(\rho,\varphi)$ is independent of φ if the phase of the field (θ) is completely random, and vice versa. We will return to Eq. (25) below.

Rewriting Eq. (3) in polar coordinates, and assuming $C(\rho,\varphi)=C(\rho)$,

$$G(\gamma)=\frac{1}{\pi\gamma}\int_0^\infty d\rho\,\rho\int_0^{2\pi}d\varphi\exp\left[\frac{-\rho^2}{\gamma}\right]C(\rho) \; , \tag{26}$$

we can carry out the integration over φ immediately. If the field is not assumed to have random phase, then $C(\rho)$ is replaced with the phase-averaged characteristic function,

$$\bar{C}(\rho)\equiv\frac{1}{2\pi}\int_0^{2\pi}C(\rho,\varphi)d\varphi \; .$$

Hence, in any case, we can always solve for the phase-averaged quantum characteristic function by the method presented in this section. This also points out the fact that the photon-counting statistics depend only on the phase-averaged quantum characteristic function.

Changing variables to $t=\rho^2$, we have

$$G(\gamma)=\frac{1}{\gamma}\int_0^\infty \exp\left[\frac{-t}{\gamma}\right]\bar{C}(\sqrt{t}\,)dt \; . \tag{27}$$

If we define the function $\tilde{C}(t)\equiv\bar{C}(\sqrt{t}\,)$, the variable $s=1/\gamma$, then we can express Eq. (27) as the Laplace Transform $(\mathcal{L}_s)$ of $\tilde{C}(t)$, evaluated at $1/\gamma$, i.e.,

$$s^{-1}G(s^{-1})=\mathcal{L}_s\{\tilde{C}(t)\} \; . \tag{28}$$

Hence, to invert this we need to evaluate the inverse Laplace transform $(\mathcal{L}_t^{-1})$ of $s^{-1}G(s^{-1})$. Finally, evaluating $\mathcal{L}_t^{-1}$ at $t=\rho^2=|\xi|^2$ and replacing $\tilde{C}$ with the phase-averaged characteristic function $\bar{C}(|\xi|)$ we have the main result of this section,

$$\bar{C}(|\xi|)=\mathcal{L}_t^{-1}\left\{\frac{1}{s}G\left[\frac{1}{s}\right]\right\}\Bigg|_{t=|\xi|^2} \; . \tag{29}$$

The inverse Laplace transform on the right-hand side can, in general, be written[15]

$$\mathcal{L}_t^{-1}\left\{\frac{1}{s}G\left[\frac{1}{s}\right]\right\}=\int_0^\infty J_0(2\sqrt{tu}\,)\mathcal{L}_u^{-1}\{G(s')\}du \; , \tag{30}$$

where J_0 is the zeroth-order Bessel function. Hence, setting $t=|\xi|^2$,

$$\bar{C}(|\xi|)=\int_0^\infty J_0(2|\xi|\sqrt{u}\,)\mathcal{L}_u^{-1}\{G(s')\}du \; . \tag{31}$$

Either Eq. (29) or Eq. (31) may be more convenient in a given application.

The form of Eq. (31) has a simple interpretation for classical fields (i.e., those with a positive P representation). From the well-known relation between the generating function for the *photon-counting* statistics and the characteristic function for the integrated intensity[16] we can write [cf. Eq. 6, above]

$$G(\gamma)=\langle e^{-\eta\gamma W}\rangle \; , \tag{32}$$

where η is the photoefficiency of the photoelectric detector and $W=\int_t^{t+\Delta t}I(t')dt'$ is the integrated field intensity. Now, for classical fields the right-hand side of Eq. (32) is just the Laplace transform of the probability density function for W, $p_W(W)$, evaluated at $s=\eta\gamma$. The inverse Laplace transform of $G(\gamma)$ is therefore (for simplicity we set $\eta=1$ so the field statistics are the same as the photon-counting statistics),

$$\mathcal{L}_u^{-1}\{G(s')\}=p_W(u) \; . \tag{33}$$

Hence, Eq. (31) applied to classical fields reduces to

$$\bar{C}(|\xi|)=\int_0^\infty J_0(2|\xi|\sqrt{u}\,)p_W(u)du \; . \tag{34}$$

Now, the characteristic function for a phase-averaged coherent state [averaged over the phase angle θ, where $\alpha=r_0\exp(i\theta)$] or, equivalently, the phase-averaged $\bar{C}(\xi)$ (averaged over φ, where $\xi=\rho\exp(i\varphi)$, can be written

$$\int_0^{2\pi}C(\xi,\xi^*)\frac{d\theta}{2\pi}=\int_0^{2\pi}e^{\xi\alpha^*-\xi^*\alpha}\frac{d\theta}{2\pi}=J_0(2|\xi|r_0) \; , \tag{35}$$

where we used Bessel's integral representation for J_0. Hence, Eq. (34) is just the average over the (integrated) field intensity (or $\hat{n}=|\alpha|^2$) of the phase-averaged quantum characteristic function for the coherent state. This relation could also have been obtained directly from Eq. (24) in the P representation by averaging over θ, changing variables to ρ^2, etc. However, Eq. (31) is more generally valid, not depending on the field being classical or even on the existence of a valid P representation. It depends only on the existence of the required inverse Laplace transforms.

IV. SOME EXAMPLES OF THE INVERSION TECHNIQUE

A. $G(\gamma)$ for a Bose-Einstein distribution

The generating function for a geometric distribution (i.e., Bose-Einstein statistics appropriate for a chaotic or thermal field) with mean photon number $\bar{n}$ is well known to be

$$G(\gamma)=\frac{1}{1+\bar{n}\gamma}\ , \tag{36}$$

yielding

$$s^{-1}G(s^{-1})=\frac{1}{s+n}\ . \tag{37}$$

The inverse Laplace transform of this quantity is quite standard, and upon replacing t with $|\xi|^2$ we have the well-known result for a Gaussian field distribution,

$$C(\xi,\xi^*)=\exp(-\bar{n}\,|\xi|^2)\ . \tag{38}$$

B. $G(\gamma)$ for Poisson photon statistics

The generating function for a Poisson distribution is

$$G(\gamma)=\exp(-\bar{n}\gamma)\ . \tag{39}$$

Hence,

$$\frac{1}{s}G\left[\frac{1}{s}\right]=\frac{1}{s}\exp\left[-\bar{n}\frac{1}{s}\right]\ , \tag{40}$$

whose inverse Laplace transform is the zeroth-order Bessel function, $J_0(2\sqrt{\bar{n}t}\,)$. Hence, using Eq. (29) and substituting $t=|\xi|^2$, we have the phase-averaged characteristic function corresponding to a Poisson photon distribution,

$$\bar{C}(\,|\xi|\,)=J_0(2\,|\xi|\sqrt{\bar{n}}\,)\ . \tag{41}$$

As we showed above, this can be interpreted either as the characteristic function for a phase-averaged coherent state [$P(\alpha)$ circularly symmetric] or as the phase-averaged characteristic function of an ordinary coherent state [$P(\alpha)=\delta^{(2)}(\alpha-\alpha_0)$].

C. $G(\gamma)$ of the general form: $1/(1+\bar{n}_T\delta)G_0[\gamma/(1+\bar{n}_T\gamma)]$

In this case we have,

$$\frac{1}{s}G\left[\frac{1}{s}\right]=\frac{1}{s+\bar{n}_T}G_0\left[\frac{1}{s+\bar{n}_T}\right]\ . \tag{42}$$

Now use the well-known property of Laplace transforms

$$\mathcal{L}_s\{\exp(at)F(t)\}=\mathcal{L}_{s-a}\{F(t)\} \tag{43}$$

to rewrite the right-hand side of the equation

$$\mathcal{L}_t^{-1}\left\{\frac{1}{s}G\left[\frac{1}{s}\right]\right\}=\mathcal{L}_t^{-1}\left\{\frac{1}{s+\bar{n}_T}G_0\left[\frac{1}{s+\bar{n}_T}\right]\right\}\ , \tag{44}$$

thus yielding

$$\mathcal{L}_t^{-1}\left\{\frac{1}{s}G\left[\frac{1}{s}\right]\right\}=\exp(-\bar{n}_Tt)\mathcal{L}_t^{-1}\left\{\frac{1}{s}G_0\left[\frac{1}{s}\right]\right\}\ . \tag{45}$$

Hence, letting $t=|\xi|^2$ and using Eq. (29) twice, we see that

$$\bar{C}(\,|\xi|\,)=\exp(-\bar{n}_T\,|\xi|^2)\bar{C}_0(\,|\xi|\,)\ . \tag{46}$$

This is seen to be the characteristic function for thermal noise added to an arbitrary, independent (phase-averaged) field. That this interpretation is correct is also shown in Appendix A in an inverse derivation using Eq. (3).

D. K-distributed noise

A type of noise with amplitude distributions based on the modified Bessel functions, $K_{\alpha-1}$, has been found useful in describing light (and radar signals) scattered from or through turbulent media.[17] The distribution of the classical amplitude A is derived from the random superposition of N individual components. However, N is a random variable having a negative binomial distribution appropriate to random scattering elements whose number evolves according to a type of birth-death–immigration process. The resulting amplitude distribution is

$$p(A)=\frac{2b}{\Gamma(\alpha)}\left[\frac{bA}{2}\right]^{\alpha}K_{\alpha-1}(bA)\ , \tag{47}$$

where $b=2\sqrt{\alpha/\langle A^2\rangle}$ and α is a parameter from the original negative binomial distribution.

Equation (47) is obtained in a completely classical derivation as the Bessel transform of the limiting form of the classical characteristic function,

$$\lim_{\bar{N}\to\infty}\langle\exp\{i(u_1\mathcal{E}_1+u_2\mathcal{E}_2)\}\rangle=\left[1+\frac{u^2\langle a^2\rangle}{4\alpha}\right]^{-\alpha}\ , \tag{48}$$

where $\mathcal{E}_1$, $\mathcal{E}_2$ and u_1, u_2 are the real and imaginary parts of the electric field and $\mathbf{u}$, respectively, $u=|\mathbf{u}|$, and $A=|\mathcal{E}|$.

We now show that, starting from the photon-number statistics implied by K-distributed noise, the quantum characteristic function obtained with our method is identical in form with that derived directly in the classical analysis. The K-distributed amplitudes have been shown to give the normalized intensity moments,

$$n_r\equiv\frac{\langle I^r\rangle}{\langle I\rangle^r}=\frac{r!\Gamma(r+\alpha)}{\alpha^r\Gamma(\alpha)}\ . \tag{49}$$

These are, in general, equal to the normalized photon factorial moments, $[\langle n(n-1)\cdots(n-r+1)\rangle]/\langle n\rangle^r$. Hence, the factorial moments are

$$\langle n(n-1)\cdots(n-r+1)\rangle=\langle n\rangle^r\frac{r!\Gamma(r+\alpha)}{\alpha^r\Gamma(\alpha)}\ . \tag{50}$$

Now, writing the generating function in terms of the factorial moments

$$G(\gamma)=\sum_{r=0}^{\infty}\frac{(-\gamma)^r}{r!}\langle n(n-1)\cdots(n-r+1)\rangle\ , \tag{51}$$

we have

$$G(\gamma)=\sum_{r=0}^{\infty}(-\gamma\langle n\rangle)^r\frac{\Gamma(r+\alpha)}{\alpha^r\Gamma(\alpha)}\ . \tag{52}$$

Taking the inverse Laplace transform of $1/sG(1/s)$ term by term we obtain

$$C(\,|\xi|\,)=\left[1+\frac{\langle n\rangle|\xi|^2}{\alpha}\right]^{-\alpha}\ . \tag{53}$$

This is seen to be exactly the same expression as Eq. (48) with $u \to |\xi|$ and $\langle n \rangle = \langle a^2 \rangle /4$.

V. THE GENERATING FUNCTION FOR SUPERPOSED FIELDS IN TERMS OF G_1 AND G_2

When two types of independent classical particles with generating functions $G_1(\gamma)$ and $G_2(\gamma)$ are added together the total particle number is simply the sum of the individual random numbers, hence it has the generating function

$$G(\gamma) = G_1(\gamma) G_2(\gamma) . \tag{54}$$

For quantum particles interference effects prohibit taking this product, although the quantum characteristic functions for independent fields may still be simply multiplied together. We now derive a formula which generalizes the classical result to account for the interference effects when two stationary, independent boson fields are superposed.

We assume that sufficient information is available for the individual fields to enable us to obtain $G_1(\gamma)$ and $G_2(\gamma)$. Using Eq. (29) we can then calculate their respective quantum characteristic functions, $C_1(|\xi|)$ and $C_2(|\xi|)$. Note that we specifically assume each field is randomly phased, hence Eq. (29) gives us the full characteristic function for each, rather than the phase averaged $\bar{C}$. We now multiply the two together to obtain the characteristic function for the superposition of the two fields (cf. Appendix B), and insert in Eq. (3) to obtain the generating function for the total field

$$G(\gamma) = \frac{1}{\pi\gamma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\frac{-|\xi|^2}{\gamma}\right] \times C_1(|\xi|) C_2(|\xi|) d^2\xi . \tag{55}$$

Making the change of variables to $\xi = \rho \exp(i\varphi)$ and $t = \rho^2$ as before and carrying out the integration over φ we obtain

$$G(\gamma) = \frac{1}{\gamma} \int_0^{\infty} \exp\left[\frac{-t}{\gamma}\right] C_1(\sqrt{t}) C_2(\sqrt{t}) dt . \tag{56}$$

This is the Laplace transform of the product of two functions for which there exists the general formula (not the usual convolution theorem),

$$\mathcal{L}_s\{f_1(t) f_2(t)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g_1(u) g_2(s-u) du , \tag{57}$$

where g_1 and g_2 are the Laplace transforms of f_1 and f_2, respectively. The path of integration is parallel to the imaginary axis and lies in a certain vertical strip in the complex u plane to the right of all poles of $g_1(u)$ and to the left of all poles of $g_2(s-u)$, and not, as indicated in published tables of the Laplace transform,[18] to the right of all poles of the integrand. Because this fact is crucial to the correct evaluation of the integral, we derive Eq. (57) along with the correct path of integration in Appendix C.

Now, with the Laplace transforms of $C_1(\sqrt{t})$ and $C_2(\sqrt{t})$ from Eq. (29),

$$G(\gamma) = \frac{1}{\gamma} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u} G_1\left[\frac{1}{u}\right] \frac{1}{s-u} \times G_2\left[\frac{1}{s-u}\right] du \bigg|_{s=1/\gamma} ; \tag{58}$$

hence, we obtain,

$$G(\gamma) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u} G_1\left[\frac{1}{u}\right] \frac{1}{1-\gamma u} G_2\left[\frac{\gamma}{1-\gamma u}\right] du . \tag{59}$$

Note that, because only the relative phase is meaningful, it is sufficient that only one of the two fields have a completely random phase. Whatever the distribution of phase for the second field, the relative phase will still be uniformly distributed in $[0, 2\pi]$.

To illustrate the use of this formula we first apply it to the addition of thermal noise to an arbitrary field, which is derived in Appendix A using Eq. (3). For this case, $G_1(\gamma)$ is that of a Bose-Einstein distribution and $G_2 = G_0$ is arbitrary. Equation (59) gives us

$$G(\gamma) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u + \bar{n}_T} \frac{1}{1-\gamma u} G_0\left[\frac{\gamma}{1-\gamma u}\right] du . \tag{60}$$

This can be evaluated using the theory of residues and completing the path of integration with a semicircle around the left half of the complex u plane. The vertical path is chosen to the right of the pole of the first factor in the integrand and to the left of any poles of the remaining portion of the integrand. Using standard techniques, it is easily shown that the contribution to the integral from the semicircle is zero. Hence, $G(\gamma)$ is equal to the sum of the residues at the poles contained within the path of integration. For this case, there is only one (simple) pole within the contour, at $u = -\bar{n}_T$. The residue is equal to

$$G(\gamma) = \frac{1}{1 + \bar{n}_T \gamma} G_0\left[\frac{\gamma}{1 + \bar{n}_T \gamma}\right] , \tag{61}$$

which agrees with the previously obtained result.

Another example using Eq. (59) is the addition of a single photon to an arbitrary field (with a nonsingular P representation) having generating function $G_0(\gamma)$. For this case substitute

$$G_1(\gamma) = (1-\gamma) \tag{62}$$

into Eq. (59) to give

$$G(\gamma) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u} \frac{u-1}{u} \frac{1}{1-\gamma u} G_0\left[\frac{\gamma}{1-\gamma u}\right] du . \tag{63}$$

The pole of g_1 is at $u = 0$, and is of order 2. The residue is therefore given by

$$G(\gamma) = \frac{d}{du}\left[(u-1) \frac{1}{1-\gamma u} G_0\left[\frac{\gamma}{1-\gamma u}\right]\right]\bigg|_{u=0} . \tag{64}$$

Carrying out the indicated differentiation and evaluating at $u=0$, we have

$$G(\gamma)=G_0(\gamma)-\gamma\frac{d}{d\gamma}\gamma G_0(\gamma)\ . \tag{65}$$

This agrees with the result we obtained using Eq. (3) in Sec. II, where we assumed the corresponding characteristic function was known. Again, caution must be exercised in its use with nonclassical fields.

VI. DISCUSSION

We have presented a number of new results for two standard tools of quantum optics, the quantum characteristic function and the photon-number generating function. The physical superposition of independent Boson fields was shown to be correctly described by multiplication of their individual characteristic functions for a wide (but not exhaustive) class of physically interesting cases. These physically describe the partial overlap of independent converging fields and can lead to a growing field intensity. The techniques developed here also allow an identification of the effect of quantum-mechanical interference for the addition of a single photon to an optical field. This is elaborated in a companion paper.[13] After showing how to invert our formula giving $G(\gamma)$ in terms of $C(\xi)$, we derived an expression for the probability generating function of a resultant superposed field directly in terms of the individual generating functions for two arbitrary fields. This generalizes the classical multiplication law for generating functions to the quantum domain.

An alternative superposition of two or more independent fields may be performed using partially silvered mirrors. Generally this process does not lead to a net increase in field intensity. Equation (18b) and a number of equations derived in Appendix B apply to this case. All of the equations derived following Eq. (18b) can be similarly modified to apply to this superposition process by suitable inclusion of reflection and transmission coefficients. This physical situation does not in general lead to an average increase in the field intensity.

Applications of the techniques to some standard problems such as the addition of thermal noise to optical fields and a model of laser propagation through the turbulent atmosphere illustrate use of the methods. Other applications of our results to understanding the evolution of the quantum statistics of light and the nature of stimulated and spontaneous emission, and to extended quantum theories of laser scattering in the atmosphere are presented elsewhere.[19] Present efforts to extend this work include applying it to two-photon spontaneous emission and squeezed states, and will be reported later.

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APPENDIX A: THE CHANGE IN $G(\gamma)$ FROM ADDED THERMAL NOISE

The characteristic function for a single-mode field consisting of thermal noise with mean photon number $\bar{n}_T$ (and having a Bose-Einstein distribution) is

$$C_T(\xi,\xi^*)=\exp(-|\xi|^2\bar{n}_T)\ . \tag{A1}$$

If this noise is added to an arbitrary field having generating function $G_0(\gamma)$ and characteristic function $C_0(\xi,\xi^*)$, the characteristic function for the total field is the product of the two,

$$C(\xi,\xi^*)=\exp(-|\xi|^2\bar{n}_T)C_0(\xi,\xi^*)\ . \tag{A2}$$

To find the generating function for the composite field use Eq. (A2) in Eq. (3),

$$G(\gamma)=\frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-|\xi|^2}{\gamma}\right]\exp(-|\xi|^2\bar{n}_T)\times C_0(\xi,\xi^*)d^2\xi\ . \tag{A3}$$

Now, define

$$1/\gamma'=1/\gamma+\bar{n}_T=\frac{1+\gamma\bar{n}_T}{\gamma}\ , \tag{A4}$$

and Eq. (A3) becomes

$$G(\gamma)=\frac{\gamma'}{\gamma}\frac{1}{\pi\gamma'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-|\xi|^2}{\gamma'}\right]C_0(\xi,\xi^*)d^2\xi\ , \tag{A5}$$

hence,

$$G(\gamma)=\frac{1}{1+\bar{n}_T\gamma}G_0\left[\frac{\gamma}{1+\bar{n}_T\gamma}\right]\ . \tag{A6}$$

This generalizes a well-known result[20] for the superposition of a coherent signal with Gaussian noise, where $G_0(\gamma)=\exp(-\gamma\bar{n})$.

APPENDIX B: THE VALIDITY OF MULTIPLYING QUANTUM CHARACTERISTIC FUNCTIONS

Because the multiplication of the characteristic functions of independent processes to obtain the characteristic function of the sum of the processes is such a standard practice, we motivate our discussion with a counterexample. The quantum characteristic function for a single-photon state is derived in Sec. II,

$$C_1(\xi,\xi^*)=1-|\xi|^2\ . \tag{B1}$$

One might have assumed that the characteristic function for the random superposition of two independent single-photon states would be the square of Eq. (B1),

$$C(\xi,\xi^*)=\{1-|\xi|^2\}^2\ . \tag{B2}$$

However, this result does not lead to all ρ_n's between zero and one. To see this, use Eq. (51) to express the probability generating function in terms of the factorial moments,

$$G(\gamma)=1-\gamma\langle n\rangle+\tfrac{1}{2}\gamma^2\langle n(n-1)\rangle\ , \tag{B3}$$

where $\langle n \rangle = 2$ and $\langle n(n-1) \rangle = 4$ are obtained from Eq. (B2) in the usual way, and all higher factorial moments are zero. Expressing the result in terms of $(1-\gamma)$, we have

$$G(\gamma) = 1 - 2(1-\gamma) + 2(1-\gamma)^2 . \tag{B4}$$

From the definition of $G(\gamma)$, Eq. (1), we obtain the apparent photon number "probabilities," $\rho_0 = 1$, $\rho_1 = -2$, and $\rho_2 = 2$. These are obviously nonsense, and not probabilities at all. It is clear that for such extremely nonclassical fields superposition is not obtained merely by multiplying the characteristic functions (we argue below that the actual problem is that such fields cannot be perfectly superposed). This analysis is easily generalized to n-photon eigenstates. However, if the final generating function is chosen to be Eq. (B4) with probability p and equal to $1-\gamma$, corresponding to one photon, with probability $(1-p)$, it is apparent that p may always be chosen small enough so that we obtain a valid generating function. This generalizes to the n-photon Fock state in a straightforward manner. Hence, if the addition of photons is a stochastic process continuous in time we have no problem.

The origin of the apparent problem above is in the operational meaning of how one physically superposes two independent quantum fields. Basically, one can either (1) combine two fields through a partially silvered mirror or (2) cause the two propagating fields to overlap within some finite spatial region. In the former case, the combined field annihilation operator a can be written as a linear superposition of that for each of the original fields: $a = \delta a_1 + \epsilon a_2$. Because the reflection and transmission coefficients satisfy $|\delta|^2 + |\epsilon|^2 = 1$, a will obey the canonical commutation relations $[a, a^\dagger] = 1$ if a_1 and a_2 do. Substituting this form of a in the definition of $c(\xi)$, Eq. (2), leads to

$$C(\xi) = C_1(\xi\delta^*)C_2(\xi\epsilon^*) ,$$

and to Eq. (18b) using the same steps which led to Eq. (18a). Repeating the derivation leading to Eq. (B4), but including the factors of δ and ϵ, now avoids the apparent problem with unphysical probabilities. Note that, in the limit as $\delta \to 1$ and $\epsilon \to 0$, $C_2(\ )$ represents an extremely attenuated version of the original field. Such an attenuated field (the part having passed through the mirror) could conceivably then be coherently amplified (without practical limitation) and hence could not be an n-particle Fock state.[21] On the other hand, $C_1(\ )$ represents an arbitrary field. However, this means of superposing fields does not lead to a net increase in field intensity with an increasing number of contributions.

On the other hand, fields which build up by independent contributions converging on a common spatial volume will lead to a net growth in the total field intensity. It is this situation which is characterized by simply multiplying the characteristic functions of the individual contributions. However, because of the fact that only part of each mode volume (of the independent contributions) is within the overlapping region, the characteristic function for the elements of field actually superposing in this way cannot be that for an n-photon Fock state.

We now show that for fields with a nonsingular P representation it is justified to multiply characteristic functions. In fact, following Glauber,[22] the P representation for the superposition of two independently generated fields is just the convolution of the individual $P(\alpha)$'s, just as would be the case if the latter were classical probability density functions,

$$P(\alpha) = \int\int P_1(\alpha - \alpha')P_2(\alpha')d^2\alpha' . \tag{B5}$$

For such states of the field, the (normally ordered) quantum characteristic function is expressed as

$$C(\xi) = \int\int e^{\xi\alpha^* - \xi^*\alpha}P(\alpha)d^2\alpha , \tag{B6}$$

and a standard calculation yields

$$C(\xi) = C_1(\xi)C_2(\xi) . \tag{B7}$$

Hence, a sufficient condition for the multiplication of characteristic functions is that the independent fields possess valid P representations. That this analysis does not apply when both fields are photon-number eigenstates is not surprising as the latter have highly singular P representations expressed as derivatives of Dirac δ functions. In fact, we will show it is sufficient for only one of the fields to have a nonsingular P representation. We assume that this is satisfied for the "arbitrary" fields dealt with in this paper, having a characteristic function, $C_0(\xi)$ because it corresponds to either of the above methods of superposition. Alternatively, one could retain the factors of δ and ϵ resulting from superposition through partially silvered mirrors.

We wish to investigate the conditions on $P_0(\alpha)$ for the original field such that the operation of adding a single photon can be described by

$$C(\xi) = [1 - |\xi|^2]C_0(\xi, \xi^*) . \tag{B8}$$

Using Eq. (B6) to express C_0 in terms of P_0, we have

$$C(\xi) = [1 - |\xi|^2]\int\int P_0(\alpha)e^{\xi\alpha^*}e^{-\xi^*\alpha}d^2\alpha \tag{B9}$$

or

$$C(\xi) = \int\int P_0(\alpha)\left[1 + \frac{d}{d\alpha}\frac{d}{d\alpha^*}\right]e^{\xi\alpha^*}e^{-\xi^*\alpha}d^2\alpha . \tag{B10}$$

Integrating by parts and, for convenience, assuming that $P(\alpha)$ is zero in a neighborhood of the origin so no terms arise from the limits of integration, we have

$$C(\xi) = \int\int e^{\xi\alpha^*}e^{-\xi^*\alpha}\left[1 + \frac{d}{d\alpha}\frac{d}{d\alpha^*}\right]P_0(\alpha)d^2\alpha . \tag{B11}$$

Hence, we see that the addition of a single photon can be represented by the change in $P_0(\alpha)$ given by

$$P(\alpha) = \left[1 + \frac{d}{d\alpha}\frac{d}{d\alpha^*}\right]P_0(\alpha) . \tag{B12}$$

Now, if P_0 is nonsingular and vanishes in a neighborhood of the origin, the same will hold for $P(\alpha)$. Hence, adding a single-photon Fock state can be represented by either Eq. (B8) or Eq. (B12). Note that if the probability of single-photon addition is $A\,\Delta t$, Eq. (B12) leads to the Fokker-Planck equation describing spontaneous emission[9,12] (cf. Eq. 23).

Equations (B8) and (B12) can be iterated in the same way as we iterated Eq. (18a). Hence, if G_m has the same meaning as in Sec. II we obtain for the characteristic function,

$$C(\xi)=G_m(|\xi|^2)C_0(\xi)\ , \tag{B13}$$

and for $P(\alpha)$,

$$P(\alpha)=G_m\left[-\frac{d}{d\alpha}\frac{d}{d\alpha^*}\right]P_0(\alpha)\ . \tag{B14}$$

APPENDIX C: THE INTEGRATION PATH FOR EVALUATING EQ. (59)

We will derive the Laplace transform for the product of two functions to obtain the correct path of integration in the complex u plane. First, define the Laplace transform,

$$g(s)\equiv\mathcal{L}_s\{f(t)\}=\int_0^\infty e^{-st}f(t)dt\ , \tag{C1}$$

where $s>c_1$, the latter known as the abscissa of convergence for the particular function.[23] The Fourier-Mellin inversion integral expressing $f(t)$ in terms of $g(s)$ is

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}g(s)ds\ , \tag{C2}$$

where the path of integration in the complex s plane is a vertical line to the right of the abscissa of convergence, i.e., $c>c_1$. Since all poles of the integrand lie to the left of this line, by Cauchy's theorem the path may be distorted to the right without changing the value of the inversion integral. Now, in the Laplace transform of the product of two functions, express the first by means of Eq. (C2),

$$\int_0^\infty e^{-st}f_1(t)f_2(t)dt =\int_0^\infty e^{-st}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{ut}g_1(u)f_2(t)du\,dt\ , \tag{C3}$$

where $c>c_1$ and $s>c_T$, the latter being the abscissa of convergence of the integrand on the left-hand side. We can now interchange the orders of integration by the assumed uniform convergence of the integrals,

$$\int_0^\infty e^{-st}f_1(t)f_2(t)dt =\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g_1(u)\int_0^\infty e^{-(s-u)t}f_2(t)dt\,du\ . \tag{C4}$$

Now carry out the indicated Laplace transform of f_2 on the right to obtain

$$\int_0^\infty e^{-st}f_1(t)f_2(t)dt=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g_1(u)g_2(s-u)du\ , \tag{C5}$$

where $s-u>c_2$, and c_2 is the abscissa of convergence for f_2. The last inequality can be rewritten as

$$u<s-c_2\ , \tag{C6}$$

which, along with

$$u>c_1\ , \tag{C7}$$

defines the path of integration, i.e., the vertical strip defined by

$$c_1<c<s-c_2\ . \tag{C8}$$

We assume that this set is nonempty, and interpret it as saying the path of integration in the complex u plane lies to the right of all poles of $g_1(u)$ and to the left of all poles of $g_2(s-u)$.

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