J. Phys. A: Math. Gen., Vol. 11, No. 10, 1978. Printed in Great Britain

Calculating generating functions from characteristic functions, with application to quantum optics

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Received 23 March 1978

Abstract. A method is found for calculating the number generating function of a harmonic oscillator density matrix directly from the quantum characteristic function. This technique is used to interpret thermal noise in a model for an optical detector. The results clearly reveal an earlier misinterpretation and emphasise that the thermal noise is a statistically independent amplitude in the detector/harmonic oscillator which is superposed with an amplitude having the same statistical nature as the incident optical field.

1. Introduction

Characteristic functions and generating functions have found wide applicability in statistical physics (Klauder and Sudarshan 1968). In quantum optics they are used as tools for calculating statistical fluctuations of the optical field or other harmonic oscillators. In particular, generating functions for the photon number probabilities (diagonal elements of the density matrix) are often used to determine the photocounting statistics.

However, the generating function is not completely equivalent to the density matrix as phase information is lost. Information concerning the origin of a given field as the combination of several independent fields is often obscured. This happens because in forming a state by superposition, one must add the associated quantum mechanical amplitudes, leading to phase-dependent interference effects for the various physical quantities of interest.

The characteristic function provides an alternative description of the density matrix which retains all of the associated phase information. In particular, a characteristic function for a superposition of two independent fields will factor into the product of the characteristic functions for the independent states that have been combined. Because the generating function contains less (but sometimes more directly usable) information than the characteristic function, it is desirable to have a means of calculating the former given the latter.

The transformation from the characteristic function to its corresponding generating function presented here should also aid in the interpretation of complicated generating functions and facilitate the identification of independent processes which contribute to the result.

0305-4770/78/0010-1879$01.00

2. Calculation of the generating function from a characteristic function

In this section we derive a method for calculating the generating function corresponding to any given characteristic function. This direct transformation may be contrasted with the method used by Rockower *et al* (1978) in which the equation of motion for the density matrix is used to determine the generating function through calculations which are independent of the derivation of the characteristic function.

The number generating function for the density matrix ρ is defined by,

$$G(\gamma)=\mathrm{tr}[\rho(1-\gamma)^{a^{\dagger}a}]=\sum_{n=0}^{\infty}\rho_n(1-\gamma)^n, \tag{1}$$

where $a^{\dagger}$ and a are the creation and annihilation operators, respectively, and

$$|1-\gamma|<1.$$

The normally ordered quantum characteristic function is defined by

$$C(\xi,\xi^*)=\mathrm{tr}[\rho\exp(\xi a^{\dagger})\exp(-\xi^* a)]. \tag{2}$$

Taking the mth derivative of $C(\xi,\xi^*)$ with respect to ξ and ξ^* and evaluating at zero yields

$$(-1)^m\left(\frac{\partial^2}{\partial\xi\partial\xi^*}\right)^m C(\xi,\xi^*)\bigg|_{\xi=0}=\langle(a^{\dagger})^m a^m\rangle. \tag{3}$$

The mth derivative of $G(\gamma)$, evaluated at zero, yields the mth factorial moment,

$$(-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}\gamma}\right)^m G(\gamma)\bigg|_{\gamma=0}=\langle n(n-1)\dots(n-m+1)\rangle. \tag{4}$$

It is well known that the two expressions on the right-hand sides of equations (3) and (4) are equal (Klauder and Sudarshan 1968). Using this equality in a McLaurin series for the generating function, we obtain

$$\begin{aligned}G(\gamma)&=\sum_{m=0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}\gamma'}\right)^m G(\gamma')\bigg|_{\gamma'=0}\frac{\gamma^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\frac{\partial^2}{\partial\xi\partial\xi^*}\right)^m C(\xi,\xi^*)\bigg|_{\xi=0}\frac{\gamma^m}{m!}=\exp\left(\gamma\frac{\partial^2}{\partial\xi\partial\xi^*}\right)C(\xi,\xi^*)\bigg|_{\xi=0}.\end{aligned} \tag{5}$$

Expressing ξ in terms of its real and imaginary parts,

$$\xi=x+\mathrm{i}y \quad\text{and}\quad \xi^*=x-\mathrm{i}y,$$

we have

$$\frac{\partial^2}{\partial\xi\partial\xi^*}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

We use the following relation, easily verified by using the convolution theorem of Fourier transforms:

$$\exp\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\right)f(x)=\frac{1}{\sqrt{(2\pi\sigma^2)}}\int_{-\infty}^{\infty}\exp\left(\frac{-(x-x')^2}{2\sigma^2}\right)f(x')\,\mathrm{d}x'.$$

Making use of this twice, we obtain the general relation between the characteristic function and its generating function†

$$G(\gamma)=\frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(\frac{-|\xi|^2}{\gamma}\right)C(\xi,\xi^*)\,\mathrm{d}^2\xi. \tag{6}$$

This is the main result of this section.

3. Examples

We can apply this result to the special case of a coherent state $|\alpha\rangle$, defined by $a|\alpha\rangle=\alpha|\alpha\rangle$, for which

$$C(\xi,\xi^*)=\exp(\xi\alpha^*-\xi^*\alpha). \tag{7}$$

Using equation (6) we readily obtain

$$G(\gamma)=\exp(-\gamma n_r), \tag{8}$$

where $n_r=|\alpha|^2$. This is the generating function for a Poisson distribution, as expected.

If we consider a Gaussian amplitude, with mean excitation number n_T, the quantum characteristic function is well known to be

$$C(\xi,\xi^*)=\exp(-|\xi|^2 n_T). \tag{9}$$

Substituting this in equation (6), we easily obtain the generating function for the Bose–Einstein distribution,

$$G(\gamma)=(1+n_T\gamma)^{-1}. \tag{10}$$

The superposition of the above two independent amplitudes (i.e., 'signal plus noise') is represented by the product of their characteristic functions,

$$C(\xi,\xi^*)=\exp(-|\xi|^2 n_T+\xi\alpha^*-\xi^*\alpha). \tag{11}$$

Again using equation (6) along with standard integrals, we find the generating function for the superposition of a coherent 'signal' with Gaussian 'noise',

$$G(\gamma)=(1+n_T\gamma)^{-1}\exp[-\gamma n_r/(1+n_T\gamma)]. \tag{12}$$

This is also a standard result (Klauder and Sudarshan 1968, Glauber 1966)‡ although our derivation using equation (6) is perhaps simpler.

In the appendix, we apply equation (6) to a less standard situation.

4. An application to noise in a detector

In addition to its computational value, this transformation may be used to interpret complicated generating functions where the contributions of independent sources may be obscured. One such application can be made to the recent derivation by Tatarskii

† A similar calculation yields a corresponding result for the relation between the characteristic functions of x and x^2, where x is a classical random variable.

‡ Klauder and Sudarshan derive this generating function for the counting distribution for a classical signal plus noise. Glauber reaches the same result for the generating function derived from the superposition of coherent and thermal fields in the P representation.

(1974) of the generating function for a photodetector illuminated by a multimode field in an approach free from the limitations and approximations of perturbation theory.

The detector is modelled as a harmonic oscillator (following Glauber 1969) so that the task becomes the solution of the Schrödinger equation for an oscillator interacting with a multimode optical field. This leads to calculation of the probabilities that the detector/oscillator makes transitions to various coherent states.

Results are obtained for illumination of the detector by a coherent field. When the detector is initially in its ground state, interaction leads to the following generating function for the detector:

$$Q_\tau(\gamma) = \exp(-\gamma n_r), \tag{13}$$

where n_r is a function of time. This is the generating function for a Poisson distribution as expected in photodetection of coherent light (Glauber 1963). For initial thermal excitation (of mean n_T) of the detector, the result for interaction with a coherent field is

$$Q_\tau(\gamma) = (1 + n_T\gamma)^{-1} \exp[-n_r\gamma(1 + n_T\gamma)^{-1}]. \tag{14}$$

The two factors in equation (14) were identified by Tatarskii as being caused by two statistically independent sources of counts in the detector. Since the first term alone is the generating function for the thermal fluctuations in the detector without illumination ($n_r = 0$), the second term was identified as the statistically independent response of the detector to the incident field. It was concluded that the response was distorted from equation (13) by the temperature of the detector.

This interpretation can be challenged on several grounds. A factoring of the generating function does not prove the existence of independent sources. Any generating function can be separated arbitrarily into the product of a desired generating function and a residual factor. The underlying nature of the system must be examined before one can justifiably claim the existence of statistically independent sources.

The result in equation (14) can, in fact, be identified as the generating function for a mixture of Gaussian (thermal) noise and a coherent signal in the detector (see equation (12)).

In this interpretation, there are two statistically independent 'fields' which have been superposed to form the state of the detector/oscillator. The characteristic function for such a superposition is just the product of the two characteristic functions for these amplitudes separately. As shown in § 3, a generating function of the form in equation (14) readily results.

Further evidence that this is the proper interpretation is found elsewhere in Tatarskii's derivation. There he showed explicitly that the amplitude of the detector/harmonic oscillator is linear in the field and detector initial conditions. Thus the final state amplitude is the sum of a field-dependent term and a term arising from the initial state of the oscillator. This addition of amplitudes confirms that it is the factoring of the characteristic function (of the amplitude), not the number generating function, which comes from two independent contributions.

The state of the detector is thus a superposition of an amplitude with the same statistical nature as the optical signal and a thermal noise amplitude due to the finite temperature of the detector. The characteristic function is then of the form:

$$C(\xi, \xi^*) = \exp(-|\xi|^2 n_T) C_1(\xi, \xi^*).$$

The value of n_T can be determined by blocking the optical illumination. Since the moments of C_1 are uniquely determined by n_T and the measured moments of C, the photocount statistics of the optical field can be unambiguously determined.

This improved interpretation brings these results into agreement with Mollow's (1968) non-perturbative treatment of a detector model. The model differs from Tatarskii's in that a large number of harmonic oscillators make up the detector. The evolution of a field-detector system was derived to all orders of perturbation theory. Among other more formal results it was shown, consistent with equation (13), that if the field is initially in a coherent state, it remains coherent throughout its interaction with the detector.

A comparison with earlier work on a quantum mechanical forced damped harmonic oscillator interacting with the field to lowest order in perturbation theory is also possible. Carusotto (1975) and Glauber (1969) show that the characteristic function, in general, factors into three independent terms: one representing the damping of the initial state of the oscillator, a second depending on the driving field, and a third describing thermal fluctuations arising from interaction with a reservoir. The interpretation presented here for Tatarskii's non-perturbative approach indicates that the basic evolution, in terms of statistically independent amplitudes, is retained in the exact solution.

Acknowledgments

We would like to thank S R Smith for useful discussions and a critical reading of the manuscript.

Appendix

In this appendix we derive the form of the number generating function corresponding to the case in which the Hamiltonian contains terms of the form

$$\sum_{jk} [\sigma_{jk}(t)a_j^\dagger(t)a_k^\dagger(t) + \text{HC}].$$

These terms arise in models of Raman and Brillouin scattering, in simple models of parametric amplification, or if one does not make the standard rotating wave approximation in the coupling of harmonic oscillators to the electromagnetic field. One sees that the solution to the equations of motion for a_k and $a_k^\dagger$ can be written in terms of the initial values (Mollow 1967) as,

$$a_j(t) = \sum_k [u_{jk}(t)a_k + v_{jk}(t)a_k^\dagger]. \tag{A1}$$

The situation corresponding to equations (7) and (13) is that in which one of the modes (e.g., the detector), is initially in the ground state and the others are in coherent states. Using equation (A1) for $j=1$ in equation (2) and the Baker–Campbell–Hausdorff relations, we can express the characteristic function for the first mode (the detector) in terms of the initially independent characteristic functions of all of the modes. We obtain the form

$$C(\xi, \xi^*) = \langle \exp(\xi a_1^\dagger)\exp(-\xi^* a_1)\rangle = \exp(\delta^*\xi^2 + \delta\xi^{*2} - |\xi|^2|v|^2 + \beta^*\xi - \beta\xi^*). \tag{A2}$$

A similar characteristic function has been obtained for a model of parametric frequency-splitting in which the fields are initially in the vacuum state (Mollow 1973). Inserting this expression in equation (6), we obtain a photon number generating function of the form

$$G(\gamma)=[(1+|v|^2\gamma)^2-4|\delta|^2\gamma^2]^{-1/2}\exp\left(\frac{a+b\gamma+c\gamma^2+\mathrm{d}\gamma^3+e\gamma^4}{[(1+|v|^2\gamma)^2-4|\delta|^2\gamma^2]^2}\right).$$

This confirms Mollow's observation (1967) that an initially coherent state vector does not remain so in the presence of this type of coupling, and hence the photon number distribution is no longer Poisson.

References

Carusotto S 1975 *Phys. Rev.* A **11** 1407
Glauber R 1963 *Phys. Rev.* **131** 2766
—— 1966 *Physics of Quantum Electronics* eds P L Kelley, B Lax and P E Tannenwald (McGraw Hill: New York) p 807
—— 1969 *Quantum Optics: Int. School of Physics 'Enrico Fermi', Course 42* (New York: Academic) p 46
Klauder J and Sudarshan E 1968 *Fundamentals of Quantum Optics* (New York: Benjamin) pp 74, 232
Mollow B R 1967 *Phys. Rev.* **162** 1256
—— 1968 *Phys. Rev.* **168** 1896
—— 1973 *Phys. Rev.* A **8** 2684
Rockower E B, Abraham N B and Smith S R 1978 *Phys. Rev.* A **17** 1100
Tatarskii V I 1974 *Sov. Phys.-JETP* **39** 423