Quantum derivation of K-distributed noise for finite $\langle N \rangle$

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Semiclassical derivations of the fluctuations of light beams have relied on limiting procedures in which the average number, $\langle N \rangle$, of scattering elements, photons, or superposed wave packets approaches infinity. We show that the fluctuations of thermal light having a Bose–Einstein photon distribution and of light with an amplitude distribution based on the modified Bessel functions, $K_{\alpha-1}$, which has been found useful in describing light scattered from or through turbulent media, may be derived with a quantum-mechanical analysis as the superposition of a random number, N, of single-photon eigenstates with finite $\langle N \rangle$. The analysis also provides the P representation for K-distributed noise. Generalizations of K noise are proposed. The factor-of-2 increase in the photon-number second factorial moment related to photon clumping in the Hanbury Brown–Twiss effect for thermal (Gaussian) fields is shown to arise generally in these random superposition models, even for non-Gaussian fields.

INTRODUCTION

The fluctuations of light beams from spontaneous emission[1] and from laser scattering through the turbulent atmosphere[2,3] have been calculated in semiclassical analyses in which it is assumed that the average number of scattering elements, photons, or superposed wave packets approaches infinity. The individual contributions may arise from the independent decay of atoms or from the refraction and diffraction from structures at various scales in atmospheric turbulence. In each case the random superposition of light leads to constructive and destructive interference and hence to amplitude fluctuations and scintillations in the receiving aperture. The model for thermal fluctuations of light seems fairly straightforward, leading as it does to a Gaussian amplitude distribution as the average number, $\langle N \rangle$, of contributions tends to infinity. The latter is an instance of the central-limit theorem,[4] which gives conditions such that the sum of a large number of independent random variables tends to a Gaussian distribution.

However, Jakeman and Pusey[3] pointed out that there is an alternative to the usual central-limit-theorem result, showing that when the number of contributions, N, is itself a random variable having a negative binomial distribution, one obtains non-Gaussian noise in the limit. The distribution of amplitudes, A, was found to have what they termed a K distribution, which is related to the modified Bessel functions. Various generalizations[5] of this model have also proven useful by extending the range over which the data on propagation of light through turbulence can be fitted to the models.

It is always of interest to determine the minimum assumptions required to arrive at models, especially for those models proving useful in describing empirical data. We show that taking the limit $\langle N \rangle \to \infty$ is not necessary in a corresponding quantum theory. In fact, the quantum formulation of this model suggests a number of paths for generalization. One such path leads to the inclusion of a deterministic coherent component to the field, similar to the I-K distribution of Phillips and Andrews.[5]

With our general formulation of the random superposition of photon eigenstates, we find that if the number of elementary sources is Poisson, as is appropriate for spontaneous emission or laser scattering from a rotating ground glass, the final field has a Gaussian amplitude and a Bose–Einstein photon-number distribution. When the number of sources has a negative binomial distribution, the final field is found to have the so-called K distribution. In fact, we obtain the P representation for K-distributed noise, which then leads to exactly the same probability-density function (PDF) for the amplitude as formulated by Jakeman and Pusey.

More generally still, we show that, whatever the distribution of elementary sources, scatterers, etc., the second factorial moment of the photon number in the field (hence $\langle I^2 \rangle$) has exactly twice the value of that for corresponding classical particles. We thus see that this famous twofold increase (related to photon bunching) measured in the Hanbury Brown–Twiss effect[6] arises quite generally in the random superposition of independent contributions to the light field, even when the resulting field is non-Gaussian.

QUANTUM THEORY OF K NOISE

A type of noise with amplitude distributions based on the modified Bessel functions, $K_{\beta-1}$, has been found useful in describing light (and radar signals) scattered from or through turbulent media. The distribution of the classical amplitude, A, is derived from the random superposition of N individual components with an average amplitude squared $\langle \epsilon^2 \rangle$. Moreover, N is a random variable having a negative binomial distribution appropriate to random scattering elements whose number evolves according to a type of birth–death–immigration process. The resulting amplitude distribution is

$$p(A) = \frac{2b}{\Gamma(\beta)} \left(\frac{bA}{2} \right)^{\beta} K_{\beta-1}(bA), \qquad (1)$$

where A is the field amplitude, $b = 2(\beta/\langle A^2 \rangle)^{1/2}$, and β is a parameter from the original negative binomial distribution.

Equation (1) was obtained in a completely classical derivation as the Bessel transform of the limiting form of the classical characteristic function,

$$\lim_{\langle N\rangle \to \infty} \langle \exp[i(u_1\mathscr{E}_1 + u_2\mathscr{E}_2)]\rangle = \left(1 + \frac{u^2\langle \epsilon^2\rangle}{4\beta}\right)^{-\beta}, \quad (2)$$

where $\mathscr{E}_1$ and $\mathscr{E}_2$ and u_1 and u_2 are the real and imaginary parts of the electric field and $\mathbf{u}$, respectively; $u = |\mathbf{u}|$; and $A = |\mathscr{E}|$.

We now derive this process in a quantum-mechanical model of scattering by random independent scattering elements. The normally ordered quantum characteristic function is defined by

$$C(\xi, \xi^*) = \mathrm{tr}[\rho \exp(\xi a^+)\exp(-\xi^* a)] = \langle \exp(\xi a^+)\exp(-\xi^* a)\rangle. \quad (3)$$

To begin, we calculate the characteristic function for the one-photon state. We insert $\rho = |1\rangle\langle 1|$ in the definition of the characteristic function and obtain

$$C_1(\xi, \xi^*) = \langle 1|\exp(\xi a^+)\exp(-\xi^* a)|1\rangle \quad (4)$$

or

$$C_1(\xi, \xi^*) = \langle 0|a \exp(\xi a^+)\exp(-\xi^* a)a^+|0\rangle, \quad (5)$$

in terms of the creation and annihilation operators a^+ and a. We use the shift property of the exponentiated operators[7] to write this as

$$C_1(\xi, \xi^*) = \langle 0|(a + \xi)(a^+ - \xi^*)|0\rangle \quad (6)$$

and, finally,

$$C_1(\xi, \xi^*) = 1 - |\xi|^2. \quad (7)$$

We first assume that there is a random number of weak scatterers in the beam propagation path, each having probability p of contributing a photon to the random superposition of scattered fields at the detector (but negligible probability of contributing two or more photons). Hence the density matrix, ρ_0, for the field contribution at the detector from an elementary scattering element somewhere on the propagation path is that for a single-photon state with probability p and zero photons with probability $(1 - p)$, i.e.,

$$\rho_0 = (1 - p)|0\rangle\langle 0| + p|1\rangle\langle 1|, \quad (8a)$$

and the corresponding characteristic function is

$$C_0(\xi, \xi^*) = (1 - p)1 + p(1 - |\xi|^2) \quad (8b)$$

or

$$C_0(\xi, \xi^*) = 1 - p|\xi|^2. \quad (9)$$

The total field is obtained as the random superposition of the contributions from each scattering element, and hence the characteristic function for the total quantum field, is the product of the characteristic functions for each independent scattering element. Subject to there being N elementary scattering elements, the total characteristic function for the field at the detector is

$$C(\xi, \xi^*)|_N = (1 - p|\xi|^2)^N. \quad (10)$$

We now average over the random variable N with a negative binomial distribution.

$$P_N = \binom{N + \beta - 1}{N} \frac{(\langle N\rangle/\beta)^N}{(1 + \langle N\rangle/\beta)^{N+\beta}}, \quad (11)$$

which is a two-parameter distribution characterized by its mean $\langle N\rangle$ and its normalized variance $1/\beta + 1/\langle N\rangle$. This distribution can be interpreted heuristically as the generalization to noninteger β of the sum of β geometric (i.e., Bose–Einstein) random variables, each of which has a mean $\langle N\rangle/\beta$ and a variance equal to $(\langle N\rangle/\beta)(\langle N\rangle/\beta + 1)$ as is appropriate for a geometric random variable. Hence the sum of β such random variables will have a mean $\langle N\rangle$ and a variance equal to $\beta\,[(\langle N\rangle/\beta)(\langle N\rangle/\beta + 1)]$. Normalizing the latter by dividing by $(\langle N\rangle)^2$ gives the required variance. The average of Eq. (10) over N is then accomplished easily by recognizing it as just the probability-generating function[6]

$$G(\gamma) = \sum_{N=0}^{\infty} P_N(1 - \gamma)^N \quad (12)$$

for a negative binomial distribution, evaluated at $p|\xi|^2$. By the heuristic interpretation of the distribution as the sum of β independent geometric random variables, it is apparent that the generating function for the number of scattering elements is that of a geometric random variable raised to the β power, i.e.,

$$G(\gamma) = \left(1 + \frac{\langle N\rangle}{\beta}\gamma\right)^{-\beta}. \quad (13)$$

Hence the characteristic function for the noise is

$$C(\xi, \xi^*) = \left(1 + \frac{\langle N\rangle}{\beta} p|\xi|^2\right)^{-\beta}. \quad (14)$$

This is of the same form as that obtained for the classical characteristic function [Eq. (2)] for the electric field in a semiclassical derivation by Jakeman and Pusey in the limit as $\langle N\rangle \to \infty$. However, no such limiting operation is needed in our quantum derivation.

Now the normally ordered quantum characteristic function for fields with a valid Glauber–Sudarshan P representation with respect to coherent states can be written as

$$C(\xi, \xi^*) = \int\int \exp(\xi\alpha^* - \xi^*\alpha)P(\alpha)\mathrm{d}^2\alpha. \quad (15)$$

Expressing ξ as $u + iv$ and α as $x + iy$, we obtain

$$C(u, v) = \int\int \exp(2ivx - 2iuy)P(x, y)\mathrm{d}x\mathrm{d}y. \quad (16)$$

This is just the Fourier transform of $P(x, y)$, and hence the inverse Fourier transform is

$$P(x, y) = \frac{4}{(2\pi)^2}\int\int \exp(2iuy - 2ivx)C(u, v)\mathrm{d}u\mathrm{d}v. \quad (17)$$

It is now convenient to use polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$, $u = \rho\cos\varphi$, and $v = \rho\sin\varphi$ and use the fact that C is a function of $\rho = |\xi|$ only. We then have

$$P(r, \theta) = \frac{4}{(2\pi)^2}\int_0^{\infty}\int_0^{2\pi} \exp[2ir\rho\sin(\varphi - \theta)]C(\rho)\rho\mathrm{d}\rho\mathrm{d}\varphi. \quad (18)$$

The integration over φ can be done by recognizing the integral representation of the Bessel function of zeroth order, J_0, giving

$$P(r) = \frac{4}{2\pi}\int_0^\infty J_0(2\rho r)C(\rho)\rho d\rho. \tag{19}$$

Note that this is not a function of θ, and hence P is circularly symmetric. Substituting our expression for $C(\rho)$, we have

$$P(r) = \frac{4}{2\pi}\int_0^\infty J_0(2\rho r)\left(1 + \frac{\langle N\rangle}{\beta} p\rho^2\right)^{-\beta} \rho d\rho. \tag{20}$$

This integral is tabulated,[8] yielding the P representation for our model,

$$P(\alpha) = \frac{\frac{4}{2\pi}\left(\frac{\beta}{p\langle N\rangle}\right)^{(1+\beta)/2} (|\alpha|)^{\beta-1} K_{\beta-1}\left[\left(\frac{\beta}{p\langle N\rangle}\right)^{1/2} 2|\alpha|\right]}{\Gamma(\beta)}, \tag{21}$$

where $K_{\beta-1}$ is a modified Bessel function of order $\beta - 1$ and where $|\alpha| = r$.

To obtain the PDF of the field amplitude, $f(A)$, we again use polar coordinates and the relation defining the amplitude PDF,

$$f(|\alpha|)d|\alpha| = \int_0^{2\pi} P(\alpha)|\alpha|d|\alpha|d\theta. \tag{22}$$

Because $P(\alpha)$ is circularly symmetric, the integration over angle is trivial, yielding

$$f(|\alpha|)d|\alpha| = 2\pi P(\alpha)|\alpha|d|\alpha|. \tag{23}$$

Now after making the replacement $|\alpha| \to A$ for the field amplitude, we have

$$f(A) = \frac{4\left(\frac{\beta}{p\langle N\rangle}\right)^{(1+\beta)/2} (A)^{\beta} K_{\beta-1}\left[\left(\frac{\beta}{p\langle N\rangle}\right)^{1/2} 2A\right]}{\Gamma(\beta)}. \tag{24}$$

With the identification of the average photon number, $p\langle N\rangle$, with $\langle A^2\rangle$ we obtain exactly the same result for the PDF of the amplitude as expressed in Eq. (1).

GENERALIZATIONS OF THE MODEL

In the foregoing analysis we assumed the probability of more than one photon from each scattering element to be negligible. We now consider a model giving rise to stronger scattering. Let the probability of one photon from each scattering element be given by p_1, and let that of two photons from each scattering element be p_2. We assume that the scattering of the two photons is incoherent; i.e., the characteristic function for the two-photon case is the square of the one-photon characteristic function[9] rather than the characteristic function of a two-photon Fock state, $|2\rangle$. Hence, after averaging over the possibilities, the characteristic function for the contribution from each scattering element is

$$C_0(\xi, \xi^*) = (1 - p_1 - p_2)1 + p_1(1 - |\xi|^2) + p_2(1 - |\xi|^2)^2 \tag{25}$$

or

$$C_0(\xi, \xi^*) = 1 - (p_1 + 2p_2)|\xi|^2 + p_2|\xi|^4. \tag{26}$$

Raising this to the Nth power and averaging N over a negative binomial distribution as before, we obtain the characteristic function in the moderate scattering regime,

$$C(\xi, \xi^*) = \left\{1 + \frac{\langle N\rangle}{\beta}[(p_1 + 2p_2)|\xi|^2 - p_2|\xi|^4]\right\}^{-\beta}. \tag{27}$$

More generally, if the probability distribution p_m of m photons scattering independently from each element into the receiver aperture has a probability-generating function $G_m(\gamma)$, we obtain, for the contribution from each scattering element,

$$C_0(\xi, \xi^*) = \sum_{m=0}^{\infty} p_m(1 - |\xi|^2)^m \tag{28}$$

or

$$C_0(\xi, \xi^*) = G_m(|\xi|^2). \tag{29}$$

Now if there are N independent scattering elements, the total characteristic function is $G_m(|\xi|^2)^N$. Averaging N over an arbitrary distribution,

$$C(\xi, \xi^*) = \sum_{N=0}^{\infty} p_N\{1 - [1 - G_m(|\xi|^2)]\}^N, \tag{30}$$

we have

$$C(\xi, \xi^*) = G_N[1 - G_m(|\xi|^2)], \tag{31}$$

where G_N is the generating function for the random variable N. For comparison, the corresponding probability-generating function for the compound classical process consisting of the sum of a random number (N) of independent identically distributed random variables (m_i) defined by

$$n = \sum_{i=0}^{N} m_i \tag{32}$$

is

$$G_{cl}(\gamma) = G_N[1 - G_m(\gamma)]. \tag{33}$$

The rth factorial moment is obtained by differentiating r times with respect to $(-\gamma)$ and then setting γ to zero. For the quantum case the factorial moments are obtained by differentiating r times with respect to ξ and r times with respect to $-\xi^*$ and then setting ξ to zero. After performing the required differentiations , we have, for the mean of the photon number n,

$$\langle n\rangle = \langle a^+a\rangle = -\frac{d^2}{d\xi d\xi^*} C(\xi, \xi^*)|_{\xi=0} = \langle N\rangle\langle m\rangle; \tag{34}$$

i.e., the average number of photons is the average number of scattering elements times the average number of photons scattered per element. The same well-known result is found for the mean of the classical process by using Eq. (33). Further differentiation of Eq. (31) yields the second factorial moment, related to the mean-square intensity, as

$$\langle n(n-1)\rangle = 2\langle N(N-1)\rangle\langle m\rangle^2 + 2\langle N\rangle\langle m(m-1)\rangle, \tag{35}$$

or, for the normalized factorial moment of the photon number and for the normalized second intensity moment,

$$\frac{\langle I^2\rangle}{\langle I\rangle^2} = \frac{\langle n(n-1)\rangle}{\langle n\rangle^2} = 2\left[\frac{\langle N(N-1)\rangle}{\langle N\rangle^2} + \frac{\langle m(m-1)\rangle}{\langle N\rangle\langle m\rangle^2}\right]. \tag{36}$$

This is exactly twice the result that we obtain for the classical process by differentiating $G_{\rm cl}(\gamma)$ twice with respect to γ. Hence we see that the factor-of-2 increase in the second factorial moment related to the Hanbury Brown–Twiss effect of photon clumping for thermal (Gaussian) light also appears more generally in random superposition models, even for non-Gaussian fields.

For spontaneous emission or the scattering of a laser beam (in approximately a coherent state) we would expect the number of elementary sources of the photons to have a Poisson distribution. Assuming that each such source contributes at most one photon, with probability p, to the receiver aperture, we have

$$G_N(\gamma) = \exp(-\langle N\rangle\gamma) \tag{37}$$

and

$$G_m(\gamma) = 1 - p\gamma. \tag{38}$$

Using these generating functions in Eq. (31), we obtain

$$C(\xi, \xi^*) = \exp(-p\langle N\rangle|\xi|^2). \tag{39}$$

This is the quantum characteristic function for a Gaussian field, which is well known to imply Bose–Einstein photon statistics, as is appropriate to a thermal (or pseudothermal) field. Note that it is obtained here without recourse to limiting procedures needed for applying the central-limit theorem.

Next, choosing a negative binomial distribution for the number of scattering elements as before, we have

$$C(\xi, \xi^*) = \left\{1 + \frac{\langle N\rangle}{\beta}[1 - G_m(|\xi|^2)]\right\}^{-\beta}. \tag{40}$$

This could form the basis for a unified approximation scheme generalizing the model of K-distributed noise, valid in both the weak and the strong scattering regimes. In general, the inversion of this characteristic function to obtain $P(|\alpha|)$ could be done numerically. In fact, only finite moments need be calculated, in general, and these may be obtained analytically from this form by repeated differentiation with respect to ξ and $-\xi^*$. This provides moments of the intensity or, equivalently, the factorial moments of the photon-number distribution. Hence we do not need the full inversion to compare any postulated form of G_m with experimental results.

By differentiating Eq. (40) we obtain the second factorial moment, related to the mean-squared intensity, as

$$\langle n(n-1)\rangle = 2(1 + 1/\beta)\langle N\rangle^2\langle m\rangle^2 + 2\langle N\rangle\langle m(m-1)\rangle \tag{41}$$

or, for the normalized factorial moments of the photon number and for the normalized second intensity moment,

$$\frac{\langle I^2\rangle}{\langle I\rangle^2} = \frac{\langle n(n-1)\rangle}{\langle n\rangle^2} = 2(1 + 1/\beta) + \frac{2}{\langle N\rangle}\frac{\langle m(m-1)\rangle}{\langle m\rangle^2}. \tag{42}$$

Note that the first term in Eq. (42) is that obtained for K noise[3]; the enhanced fluctuations represented by the second factor are nonzero only if both $\langle N\rangle$ is finite and there can be more than one photon scattered from each elementary scattering element.

One final generalization of K noise, suggested by the I–K distribution of Phillips and Andrews,[5] is to include a deterministic coherent field with the randomly scattered field. We do this by multiplying the quantum characteristic function for K noise [Eq. (14)] by the characteristic function for a coherent state, obtaining[10]

$$C(\xi, \xi^*) = \exp(\xi\alpha_0^* - \xi^*\alpha_0)\left(1 + \frac{\langle N\rangle}{\beta}p|\xi|^2\right)^{-\beta}. \tag{43}$$

The P representation corresponding to this characteristic function is simply Eq. (21) with the replacement $\alpha \rightarrow \alpha - \alpha_0$. Although this model obviously reduces to K-distributed noise when the amplitude of the coherent component is zero, as does I–K-distributed noise, it is not identical to the semiclassical model of Ref. 5. In fact, calculating the second factorial moment from Eq. (43) yields

$$\langle n(n-1)\rangle = |\alpha_0|^4 + 4p\langle N\rangle|\alpha_0|^2 + 2(p\langle N\rangle)^2\frac{\beta+1}{\beta}, \tag{44}$$

where the last term is the correct result for K-distributed noise when α_0 is zero. This is not the same as the corresponding result for I–K noise,

$$\langle I^2\rangle = A^4 + 2\frac{\beta+1}{\beta}p\langle N\rangle A^2 + 2(p\langle N\rangle)^2\frac{\beta+1}{\beta}, \tag{45}$$

except when $\beta = 1$.

DISCUSSION

The models of random superposition of independent field contributions presented here are shown to reproduce known fluctuation phenomena for thermal fields and for fields resulting from laser-light scattering in a turbulent atmosphere. By formulating the light as a quantum field, it is demonstrated that the photon-bunching characteristic of the Hanbury Brown–Twiss effect appears with exactly the same twofold increase in intensity fluctuations over a classical particle result, even when the model implies non-Gaussian fields.

Although there is an expanding body of experimental and theoretical results increasing our understanding of scintillation and laser-propagation phenomena, many of the successful theories are of a heuristic nature. By identifying the minimal assumptions necessary to fit experimental data, the modeling and understanding of such phenomena can be put on a firmer theoretical foundation. Such a formulation also suggests natural directions for generalizing existing models.

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