Statistics of the Stokes parameters for gaussian distributed fields

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We present a statistical characterization of the Stokes parameters S_j (through their first-order cumulants and probability density functions) for a set of partially polarized, quasi-monochromatic plane waves which is gaussian distributed. The coherency matrix formalism along with the geometric interpretation of the Stokes parameters provide guidance for these analytical calculations. Among many results, it is found that, except for totally unpolarized light, the probability density functions of S_j ($j=1, 2, 3$) are not generally symmetric around their means.

1. Introduction

Recently, Barakat [1] considered the time-statistical properties of the Stokes parameters for a partially polarized quasi-monochromatic light field. This exercise is important because the Stokes parameters are the observables of the plane wave field at optical frequencies. The set of these four real parameters, $\langle S_j \rangle$ ($j=0$–3), describes the second-order polarization properties of the field. A well-known geometric representation of these quantities is given by the so-called Poincaré sphere [2] (see also ref. [3], related topics have been discussed in ref. [4]) Σ_p^2 of radius P in three-dimensional space with axes $\langle S_j \rangle / \langle S_0 \rangle$, $j=1, 2, 3$. This is the set of the states of polarization P where the degree of polarization is given. Moreover, it has also been pointed out that a useful interpretation of the $\langle S_j \rangle$ is in terms of intensity differences for certain orthogonal polarization states (i.e., of opposite handedness and with major axes of their vibration ellipses which are orthogonal).

Among the implications of these insights, two are important to emphasize. Unpolarized light is the unique totally symmetric state localized at the center of Σ_p^2. On the other hand, polarization means symmetry breaking since P can also be viewed as the analogue of an order parameter [5]. For example, to right circularly polarize means to displace the state of polarization (described by a point in the Poincaré sphere Σ_p^2) towards the positive axis of $\langle S_3 \rangle / \langle S_0 \rangle$.

The first-order moments of S_j can be evaluated when the probability density function (PDF) of the transverse electric field (assumed here to be a distribution of fluctuating two-dimensional analytic signals) is known. In this respect, the results obtained by Barakat were based on the utilization of a PDF form which refers to the one given by Goodman [6] (denoted below as the gaussian–Goodman-type field). These results are appealing since one can make several comments:

(i) it is suggested that when using this type of PDF, $\langle S_3 \rangle = 0$ independently of the degree of polarization;

(ii) gaussian states would lie in the equatorial circle Σ_p^1 of the Poincaré sphere.

Among the implications of (i) and (ii), one can state the following argument: a right-circularly polarized beam of light cannot be gaussian distributed, or conversely, a gaussian distributed light field (such as blackbody thermal radiation) cannot be right circularly polarized. Clearly, this fact is in contradiction with experiments (e.g., refs. [2–4]) which leads us to believe that the origin of the difficulty lies in the definition of the gaussian–Goodman-type distribution. As it seems to us that this assumption is too restrictive for polarization optics applications, we feel

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it may be helpful to have a discussion of this problem in a more general context.

Our method will follow Barakat's work [1], but our derivation differs from ref. [1] not only in that it extends the results to a wider class of gaussian field distribution which is of most interest in optics, but also in that it is computationally more straightforward. This paper is divided into two main sections. First, we summarize in section 2 our notation and some ingredients used later. Second, we present a complete derivation of the cumulants of the Stokes parameters and PDFs in section 3, with complete calculations given in the appendices.

2. Notation

Following the usual statistical theory of partially polarized light [2–6] (specialized to a collection of quasimonochromatic plane waves), we adopt the analytical signal representation of the transverse electric field components, i.e. $E_j(\hat{\boldsymbol{R}}, t)=\bar{E}_j(\hat{\boldsymbol{R}}, t)+\mathrm{i}H(\bar{E}_j(\hat{\boldsymbol{R}}, t))$, where $j=1$, 2 labels two mutually orthogonal directions in the plane perpendicular to the wave vector. Here $\hat{\boldsymbol{R}}$ denotes the space coordinates and $H(\ )$ is the Hilbert transform. This representation is most useful because it allows us to define a one-to-one correspondence between field components and Stokes parameters. In the quasi-monochromatic description (narrow spectral range compared to ω) and at a fixed point, one has $\bar{E}_j(t)=a_j(t)\exp[\mathrm{i}(\omega t+\phi_j(t)]$. Introducing the random character of $E_j(t)$, we assume that $\bar{E}_j(t)$ [and consequently $H(\bar{E}_j(t))$] is a zero-mean stationary, gaussian random process. We have in mind the description of light such as thermal blackbody radiation [2–4,7,8]. Then, the joint bivariate (real and imaginary parts) complex PDF with zero mean is given in standard notation [7] by

$$p(E_1, E_2)=\frac{\det(A)}{(2\pi)^2}\exp\left(-\tfrac{1}{2}\sum_{i;j=1}^{2} E_i^* A_{ij} E_j\right), \quad (1)$$

where the hermitian and positive definite 2×2 matrix has the dimension of the inverse of an intensity. Physically, this matrix is simply related to the coherency matrix Φ, whose elements are

$$\Phi_{ij}=\langle E_i E_j^*\rangle=2(A^{-1})_{ij}. \quad (2)$$

To make contact with the Φ formalism [2–5] we point out here some useful properties. Let us denote by Φ_u (Φ_p) the notation for a coherency matrix when $P=0$ ($P=1$); then we recall the following properties:

$$\Phi_\mathrm{u}=\tfrac{1}{2}\langle S_0\rangle\sigma_0, \qquad \Phi_\mathrm{p}^2=\Phi_p\,\mathrm{tr}(\Phi_\mathrm{p}), \quad (3)$$

$$\Phi=\Phi_\mathrm{u}+\Phi_\mathrm{p}. \quad (4)$$

Eqs. (3) have been physically interpreted in ref. [5]. Relation (4) originates from the isotropy of Σ_p^2 (i.e. convexity of polarization states). The Stokes parameters $\langle S_j\rangle$ are obtained from Φ_{ij} with

$$\langle S_j\rangle=\mathrm{tr}(\Phi\sigma_j), \quad (5)$$

where σ_j is the usual notation for the Pauli matrices, σ_0 being the identity matrix. We also recall the decomposition theorem for Φ,

$$\Phi=\tfrac{1}{2}\sum_{j=0}^{3}\langle S_j\rangle\sigma_j. \quad (6)$$

In the linear basis with horizontal (1) and vertical (2) axes the instantaneous Stokes parameters are defined by

$$S_0=|E_1|^2+|E_2|^2,$$
$$S_1=|E_1|^2-|E_2|^2,$$
$$S_2=E_1^*E_2+E_1E_2^*,$$
$$S_3=\mathrm{i}(E_1^*E_2-E_1E_2^*). \quad (7)$$

Note that $\langle S_3\rangle=0$ requires that Φ_{12} be real, hence that Φ be a real matrix. The key point that we shall use in the remainder of this paper is to recognize that S_2 and S_3 can be derived from S_1 with a change of coordinates. This point was already noted in ref. [1], but not used in the computations of the different moments of S_j. Physically, it makes sense since it comes from the isotropy of the Poincaré sphere. These transformations are

$$S_1\xrightarrow{R_2}S_2 \quad \text{with } R_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},$$

$$S_1\xrightarrow{R_3}S_3 \quad \text{with } R_3=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}. \quad (8)$$

Observe that transformations from one pair of basis states to another are unitary 2×2 matrices; physically they are in correspondence with the rotations in Poincaré space. The operation on the coherency matrix is simply a similarly transformation $\Phi \to R_s \Phi R_s^{-1}$, which leaves its eigenvalues invariant.

3. Statistics of the Stokes parameters

From our discussion in section 2, we see that we need characterize only two random processes, S_0 and S_1. We want to calculate the three first cumulants $\chi_1(S_j)$ of S_j: $\chi_1 = \langle S_j \rangle$, $\chi_2 = \mathrm{var}(S_j)$, and χ_3, which give information about the mean of S_j, the spreading of the PDF(S_j) around its mean, and the asymmetry of the PDF(S_j) around its mean, respectively. Our strategy lies in the computation of the characteristic function of S_0 and S_1 (detailed in appendix A), which permits us also to determine the PDF(S_j) by an inverse Fourier transform (see appendix B). From the cumulant generating function, eqs. (A.4), (A.5), it follows that

$$\chi_s(S_0) = \Gamma(s)\ \mathrm{tr}(\Phi^s)\ , \quad (9)$$

$$\chi_s(S_1) = \Gamma(s)\ \mathrm{tr}((\Phi\sigma_1)^s)\ , \quad (10)$$

where $\Gamma(s)$ is the gamma function of the indicated argument. Now evaluating these quantities for arbitrary P, we obtain

$$\chi_1(S_0) = \langle S_0 \rangle\ , \qquad \chi_1(S_1) = \langle S_1 \rangle\ ,$$
$$\chi_2(S_0) = \langle S_0 \rangle^2 (1+P^2)/2\ ,$$
$$\chi_2(S_1) = \langle S_0 \rangle^2 (1-P^2)/2 + \langle S_1 \rangle^2\ ,$$
$$\chi_3(S_0) = \langle S_0 \rangle^3 (1+3P^2)/2\ ,$$
$$\chi_3(S_1) = \langle S_1 \rangle (\tfrac{3}{2}\langle S_0 \rangle^2 (1-P^2) + 2\langle S_1 \rangle^2)\ . \quad (11)$$

Eqs. (11) tell us that the variance of S_0 (S_1) depends quadratically on the degree of polarization. The asymmetry factor for S_1 vanishes only when $\langle S_1 \rangle = 0$. Notice that the results for S_0 are in agreement with those of Mandel [8] and Saleh [7]. When $P=1$, $\chi_3(S_1) = 2\langle S_1 \rangle^3$, which is positive or negative depending on the sign of $\langle S_1 \rangle$ [e.g., for $\pm 45°$ linearly polarized states, one gets $\langle S_1 \rangle = \pm \langle S_0 \rangle$, $\chi_2(S_1) = \langle S_0 \rangle^2$, $\chi_3(S_1) = \pm 2\langle S_1 \rangle^3$]. Observe from appendix B that $p(S_0)$ reduces to a negative exponential distribution when $P=1$ and that the PDF(S_1) is not symmetric around $\langle S_1 \rangle$ unless $P=0$.

Now for S_2 and S_3, we have to make use of the similarity transformation on the coherency matrix,

$$\chi_s(S_k) = \Gamma(s)\ \mathrm{tr}[(R_k \Phi R_k^{-1} \sigma_1)^s]$$
$$= \Gamma(s)\ \mathrm{tr}((\Phi\sigma_k)^s)\ , \quad \text{for } k=2, 3\ , \quad (12)$$

since $R_2^{-1}\sigma_1 R_2 = \sigma_2$ and $R_3^{-1}\sigma_1 R_3 = \sigma_3$. It follows that (for $k=1, 2, 3$)

$$\chi_1(S_k) = \langle S_k \rangle\ ,$$
$$\chi_2(S_k) \langle S_0 \rangle^2 (1-P^2)/2 + \langle S_k \rangle^2\ ,$$
$$\chi_3(S_k) = \langle S_k \rangle (\tfrac{3}{2}\langle S_0 \rangle^2 (1-P^2) + 2\langle S_k \rangle^2)\ , \quad (13)$$

which means that the S_j have the same cumulant form.

The statistics of Stokes parameters discussed above has some remarkable properties. First, note that $\chi_3(S_{j\neq 0}) \neq 0$ simultaneously for $j=1, 2, 3$ unless $P=0$. This means that S_j have generally nonsymmetric PDFs around their mean. With regard to point (i) of the introduction, we observe that we do not require that $\langle S_3 \rangle = 0$, which implies that (thermal) gaussian distributed fields are not restricted to Σ_p^1 of the Poincaré sphere. It seems very likely that the Goodman type of multivariate complex gaussian distribution (eq. (3.9) of his paper [6]) has a rather obscure significance in optics since it excludes the case of circularly polarized light, which is of great interest in many experimental situations. Note that Steeger et al. [9–11] have examined other aspects of this problem (i.e. spatial fluctuations of the Stokes parameters with application to speckle fields), which differ from the present temporal situation.

This exercise is not entirely academic. Steeger et al. have also noted that the statistics of the Stokes parameters is a topic which has a potentially wide area of application to investigate the polarization properties of scattering and radiation processes (e.g., characteristics of a random rough surface, propagation in monomode fiber optics, etc.).

It seems desirable that quantitative time-statistical analyses of the Stokes parameters should be experimentally performed since we are not able to locate such measurements in the literature to date. It would also be worth examining how these results are modified when one superposes a coherent light contribution on a partially polarized thermal beam, or for more complex field distributions (e.g. K distributions [12]).

Appendix A

This appendix deals with the method of calculation for the cumulants of S_j. The line of reasoning is similar to that of Saleh [7]. Consider the following expression for the characteristic functions of S_0 and S_1 [eqs. (A.1) and (A.2), respectively]:

$$CS_0(u)=\langle\exp(iuS_0)\rangle=\frac{\det A}{(2\pi)^2}\iint dE_1\,dE_2$$

$$\times\exp\left(-\tfrac{1}{2}\sum_{i,j=1}^{2}E_i^*(A_{ij}-2iu\delta_{ij})E_j\right)$$

$$=\det A\,[\det(A_{ij}-2iu\delta_{ij})]^{-1}$$

$$=\det A\prod_{j=1}^{2}(a_j-2iu)^{-1}$$

$$=(1-2iu/a_1)(1-2iu/a_2)\,, \tag{A.1}$$

$$CS_1(u)=\det A(a_1-2iu)^{-1}(a_2+2iu)^{-1}$$

$$=(1-2iu/a_1)(1+2iu/a_2)\,, \tag{A.2}$$

where the a_j denote the eigenvalues (supposed distinct) of A and δ_{ij} is the Kronecker symbol. If there is any degeneracy (e.g., if $P=0$), the expressions have to be modified by taking the limiting value when $a_1\to a_2$. The cumulants are the coefficients of $u^k/\Gamma(k+1)$ in a MacLaurin expansion of $\ln C(-iu)$,

$$\ln\langle\exp(uX)\rangle=\sum_{k=1}^{\infty}\frac{u^k}{\Gamma(k+1)}\chi_k(X)\,, \tag{A.3}$$

where $\Gamma(n)$ is the gamma function. This yields

$$\chi_k(S_0)=2^k\Gamma(k)\,\mathrm{tr}((A^{-1})^k)=\Gamma(k)\mathrm{tr}((\Phi)^k)\,, \tag{A.4}$$

$$\chi_k(S_1)=2^k\Gamma(k)\,\mathrm{tr}((A^{-1}\sigma_1)^k)=\Gamma(k)\,\mathrm{tr}((\Phi\sigma_1)^k)\,. \tag{A.5}$$

Appendix B

In this appendix we compute the PDF of S_0 and S_1. The general procedure is to observe that the PDF can be obtained by writing the inverse Fourier transform of the characteristic function $C(u)$. Hence, we get

$$p(S_0)=\frac{\det A}{2\pi}\int\frac{\exp(-iuS_0)\,du}{(a_1-2iu)(a_2-2iu)}$$

$$=-\frac{a_1a_2}{8\pi}\int\frac{\exp(-iuS_0)\,du}{(u+ia_1/2)(u+ia_2/2)}\,. \tag{B.1}$$

Since $S_0>0$, the integral converges and is simply evaluated by contour integration (i.e. residue theory). It follows that

$$p(S_0)=\frac{a_1a_2}{2(a_2-a_1)}$$

$$\times[\exp(-a_1S_0/2)(-\exp(-a_2S_0/2)]\,,$$

$$a_2=\frac{u}{\langle S_0\rangle(1+P)}\,,\qquad a_1=\frac{u}{\langle S_0\rangle(1-P)}\,,$$

$$a_1>a_2>0\,. \tag{B.2}$$

Hence

$$p(S_0)=\frac{1}{P\langle S_0\rangle}\left[\exp\left(-\frac{2S_0}{\langle S_0\rangle(1+P)}\right)\right.$$

$$\left.-\exp\left(-\frac{2S_0}{\langle S_0\rangle(1-P)}\right)\right]. \tag{B.3}$$

This agrees with the results of Mandel [8] and Saleh [7]. Observe that $P(S_0)$ depends only on the measurable parameters $\langle S_0\rangle$ and P. The same remark holds for degeneracy in appendix A. For S_1 we obtain

$$p(S_1)=\frac{\det A}{2\pi}\int\frac{\exp(-iuS_1)\,du}{(a_1-2iu)(a_2+2iu)}$$

$$=\frac{a_1a_2}{8\pi}\int\frac{\exp(-iuS_1)\,du}{(u+ia_1/2)(u-ia_2/2)}\,. \tag{B.4}$$

The integration in eq. (B.4) can be carried out in the same way as for eq. (B.1). After an integration over a contour for which the integral over the boundaries is found to converge (note that we have to discriminate between the cases $S_1>0$ and $S_1<0$) we get the following expressions: if $S_1<0$ then

$$p(S_1)=\frac{a_1a_2}{2(a_2+a_1)}\exp(a_2S_1/2)$$

$$=\frac{2}{\langle S_0\rangle(1-P^2)}\exp\left(\frac{S_1}{\langle S_0\rangle(1+P)}\right), \tag{B.5}$$

and if $S_1>0$ then

$$p(S_1)=\frac{a_1 a_2}{2(a_2+a_1)}\exp(-a_1 S_1/2)$$
$$=\frac{2}{\langle S_0\rangle(1-P^2)}\exp\left(-\frac{S_1}{\langle S_0\rangle(1-P)}\right),$$

which can be put in the following form:

$$p(S_1)=\frac{2}{\langle S_0\rangle(1-P^2)}\times\exp\left(-\frac{|S_1|}{\langle S_0\rangle[1-\mathrm{sign}(S_1)P]}\right), \quad \text{(B.6)}$$

with $\mathrm{sign}(u)=u/|u|$. From the same method, expressions can be obtained for $p(S_2)$ and $p(S_3)$ with corresponding modifications [Jacobian transformations of (A.1) resulting from eqs. (8)].

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