Self-similarity and long-tailed distributions in the generation of thermal light

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Two counterintuitive phenomena are studied. (1) It is well known that a thermal electromagnetic field has a Bose–Einstein (geometric) distribution of photons within a coherence volume. This arises because of the photon clumping characteristic of a thermal Boson field. On the other hand, the distribution of the number of atoms emitting photons through spontaneous emission must be Poisson if emissions are truly independent. (2) The average time between atomic decays is finite, being just the inverse of the total decay rate of the atoms. However, it is shown that in a coherence volume or in a single mode of the resulting Gaussian electromagnetic field, the average photon interarrival time is infinite. Hence, on average, an infinite length of time must pass before $\langle N \rangle$ photons arrive in the field. These apparent paradoxes are discussed, showing how both arise from random interference of Boson fields. The infinite waiting time is seen to be one manifestation of a long-tailed distribution. Such distributions are increasingly important by virtue of their relation to self-similarity and fractals, e.g., strange attractors in the description of deterministic chaos; therefore, it is of interest to understand their counterintuitive properties and see how they arise naturally even in more traditional analyses.

I. INTRODUCTION

Long-tailed distributions with infinite moments beyond a given order occasionally arise in physical models. One well-known example of such a distribution is the Lorentz (Cauchy) distribution. Students sometimes feel it somewhat paradoxical that a distribution with infinite mean or variance can have any physical significance. We show that a long-tailed distribution arises as the probability density function (pdf) for the time of arrival of photons in a coherence volume or in a single mode of a linearly growing Gaussian electromagnetic field. Such a field has a Bose–Einstein (geometric) photon number distribution and may be generated by the spontaneous emission of excited atoms or laser scattering from a rotating ground glass.[1] The distribution of photon interarrival times does not, in fact, possess even a finite mean. On the other hand, the average number of photons in the field is nonzero. In spontaneous emission, $\langle N \rangle$ equals the average number of decayed atoms. In laser light scattered from a rotating ground glass, $\langle N \rangle$ is proportional to the integrated laser field intensity. In either case $\langle N \rangle$ can be written as At.

The apparent paradox relating to infinite photon interarrival times is related to another puzzling fact. If spontaneous emissions by atoms are really independent events, then the distribution of the number of decayed atoms must be Poisson. How can this be reconciled with the well-known Bose–Einstein photon distribution of thermal light? We answer these questions by showing how both of these phenomena can be understood physically as arising from the random interference of boson fields.

In Sec. II, we present a method for finding the distribution of the time until the first photon arrives and apply it to a Poisson (independent emissions) process. Section III applies the method to the linearly growing Bose–Einstein distribution within a coherence volume of the field and shows generally that the mean time between photon arrivals is infinite. In Sec. IV, this apparent paradox is shown to be simply a manifestation of a distribution having a very slowly decreasing tail. By analyzing a simple, idealized distribution, the situation is clarified and the mathematical nature of the apparent paradox is explained. A simple physical model is presented in Sec. V, showing how random interference of boson fields leads to the long-tailed distributions of photon interarrival times in a coherence volume.

The distribution derived in Sec. III is of a type known as a hyperbolic distribution. Such distributions fall off asymptotically according to a power law. Because they are increasingly useful by virtue of their relation to self-similarity, scaling, and fractals, e.g., strange attractors in the description of deterministic chaos, it is of interest to understand the nature of their counterintuitive properties. Also, it is instructive to see how they arise naturally even in more traditional analyses. Their importance is discussed further in Sec. VI, where the self-similar nature of the evolution of thermal light is also explained.

II. PHOTON INTERARRIVAL TIMES IN A POISSON PROCESS

In order to fix notation and introduce the method of analysis we first derive the average time between atomic decays and the pdf of the time until the first decay given the distribution of the number of decays at time t. This analysis works backward compared to the usual analysis of the Poisson process. We assume that the decay of each atom is completely independent of all other atoms and the number of excited atoms is held constant by a pumping process. The probability distribution of $N(t)$, the random variable corresponding to the number of decayed atoms by time t, is Poisson:

$$\Pr[N(t)=n]\equiv p_n=\langle N\rangle^n \exp(-\langle N\rangle)/n!, \tag{1}$$

with the average (or mean)

$$\langle N\rangle = At, \tag{2}$$

where A is the total rate of atomic decay.

A. General method

If the number of atomic decays by time t, $N(t)$, is less than n, it implies that the time of occurrence of the nth decay is later than t; the same holds conversely. In other words, the event[2] $[N(t)<n]$ is equivalent to the event $(T_n>t)$, where T_n is the random variable corresponding to the time at which the nth atomic decay occurs. Hence, the corresponding probabilities of these events are equal, i.e.,

$$\Pr(T_n>t)=\Pr[N(t)<n]. \tag{3}$$

The right-hand side of Eq. (3) is (almost) the cumulative distribution function (CDF) for N; hence, we can write

$$\Pr(T_n>t)=\sum_{1=0}^{n-1} p_i. \tag{4}$$

Taking the first difference of Eq. (4) we have

$$\Pr(T_n>t)-\Pr(T_{n-1}>t)=p_{n-1}. \tag{5}$$

Now, the average ($\langle T\rangle$) of a random variable with pdf $[f(t)]$,

$$\langle T\rangle=\int_0^\infty tf(t)dt, \tag{6}$$

and CDF $[F(t)]$ [i.e., $\Pr(T\leqslant t)$],

$$F(t)=\int_0^t f(t')dt', \tag{7}$$

whose range is $[0,\infty]$, can be expressed as

$$\langle T\rangle=\int_0^\infty [1-F(t)]dt \tag{8}$$

through integration by parts of Eq. (6). Hence, integrating Eq. (5) from $0\to\infty$, we have

$$\langle T_n\rangle-\langle T_{n-1}\rangle=\int_0^\infty p_{n-1}\,dt. \tag{9}$$

B. Application to the Poisson case

The integral on the right-hand side of Eq. (9) is easily carried out using Eqs. (1) and (2) and the definition of the Γ function, yielding

$$\langle T_n-T_{n-1}\rangle=\langle T_n\rangle-\langle T_{n-1}\rangle=1/A. \tag{10}$$

The average time between decays is just the inverse decay rate, as it should be. Note that we have used the linearity of the averaging operation to write the average of the difference of two random variables as the difference of their averages.

Using Eq. (5) with n equal to 1 we find $1-F(t)$ for the time until the first decay; hence,

$$F(t)=1-p_0=1-\exp(-At) \tag{11a}$$

and, differentiating with respect to time,

$$f(t)=A\exp(-At). \tag{11b}$$

Equations (11) are the CDF and pdf, respectively, of a negative exponential random variable. In fact, it is well known that not just the time until the first decay, but all the interdecay times are exponential random variables.

III. APPLICATION TO THE BOSE–EINSTEIN CASE

We now apply the method of analysis of Sec. II to the generation of the thermal (Gaussian) field in a given coherence volume. In this case, the number of photons has the distribution[3]

$$p_n=\langle N\rangle^n/(1+\langle N\rangle)^{n+1} \tag{12}$$

in terms of the average number of photons $\langle N\rangle$. This is known as a geometric or Bose–Einstein distribution. For nonequilibrium situations such as the thermal field generated from spontaneous emission, in a linear laser amplifier with gains equal to losses,[4] or for pseudothermal light produced by laser scattering from a rotating ground glass, the average is given by $\langle N\rangle = at$. If there are m coherence volumes in the field fed by spontaneous emission, then $a=A/m$. Using Eq. (12) in Eq. (9), we have, for the average interarrival time of photons in the field,

$$\langle T_n-T_{n-1}\rangle=\langle T_n\rangle-\langle T_{n-1}\rangle=\int_0^\infty \frac{(at)^{n-1}}{(1+at)^n}dt. \tag{13}$$

The integral in Eq. (13) is seen to be logarithmically divergent for all n. In particular, the time until the first photon arrives is, on average, infinite. Again, setting n equal to 1 in Eq. (5), we obtain the CDF of the time until the first photon arrives [cf. Eq. (11)]:

$$F(t)=1-1/(1+at). \tag{14a}$$

The corresponding pdf is found by differentiation:

$$f(t)=a/(1+at)^2. \tag{14b}$$

Equations (14) are an example of what is called a Pareto distribution. Such distributions, falling off asymptotically according to a power law, are termed hyperbolic. In general, their moments beyond a given order are infinite. It is easy to see that the distribution in Eqs. (14) has infinite moments of all orders, beginning with the mean.

We thus have an apparent paradox. On one hand, the average number of photons at time t in a given mode or coherence volume of a field starting from an initial vacuum state at $t=0$ is nonzero; in fact, it is given by $\langle N \rangle = at$. On the other hand, the average time between photon arrivals is infinite.[5] Hence, on average we must wait an infinite length of time for $\langle N \rangle$ photons to arrive in the field. Stated differently, the average number of photons arriving in a finite time t *would*, on average, take an infinite length of time to arrive.

That Eqs. (1) and (12), and (11) and (14a) are different is a consequence of the random interference of the independently emitted photons, subsequent to their emission. The details of the evolution of the quantum statistics of light through random interference are not the main focus here, although we show physically in Sec. V how the boson nature of the field leads to a long-tailed distribution. Other aspects of this evolution are discussed at greater length elsewhere.[6]

IV. AN IDEALIZED LONG-TAILED DISTRIBUTION

In order to resolve the apparent paradox, we now show that it is simply one manifestation of a long-tailed distribution: We will see that although it may be counterintuitive it is not a paradox. Consider the distribution with pdf

$$f(t) = 0.618\,\delta(t-t_0) + 0.382\,\delta(t-c) \qquad (15)$$

in which $\delta(\ )$ is the Dirac delta function. We will let $c \to \infty$. Let this distribution determine the random times between the successive occurrences of some event and let $N(t)$ be the random number of events occurring in time t. We will show that this idealized distribution manifests the same apparent paradox as above. First, note that the average of the distribution,

$$\langle T \rangle = 0.618\,t_0 + 0.382c, \qquad (16)$$

goes to infinity[7] with c. Hence, the average time until the first event is infinite. However, with probability 0.618 the first event occurs at time t_0. Given that occurrence, the average time between the first and second events will be infinite, etc. On the other hand, the mean number of events occurring in a finite time t will generally be nonzero, as given by

$$\langle N(t) \rangle = \begin{matrix} 0, & \text{if } t<t_0, \\ 0.618, & t_0 \leqslant t < 2t_0, \\ 1, & 2t_0 \leqslant t < 3t_0, \\ \vdots & \vdots \end{matrix} \qquad (17)$$

Equation (17) can be determined by enumerating the various outcomes and their corresponding probabilities with a probability tree. We see that the average number of events is actually nonzero for $t \geqslant t_0$. In fact, on average one event will have occurred by time $2t_0$, even though the average time until the first event is infinite. While at first this may seem surprising, it is clearly not a paradox.

V. PHYSICAL ORIGIN OF THE LONG-TAILED DISTRIBUTION

We now present a physical picture of how the long-tailed distribution arises in the generation of thermal light. Define the rate function $r(t)$ for a process so that $r(t)dt$ is the conditional probability that an event will occur in the interval $(t, t+dt]$ given that it has not yet occurred up to time t. We can express this as

$$r(t)dt = \Pr(t < T_n \leqslant t+dt \,|\, T_n > t), \qquad (18)$$

or, using the general multiplication rule for conditional probabilities,[8]

$$r(t)dt = \frac{\Pr(t < T_n \leqslant t+dt)}{\Pr(T_n > t)} = \frac{f(t)dt}{1-F(t)}. \qquad (19a)$$

We have taken $n=1$ so that T_1 is the time of the first event, $f(t)$ is the pdf of T_1, and $F(t)$ is the corresponding CDF. To make the meaning of $r(t)$ clearer, we can integrate Eq. (19a) to obtain

$$f(t) = r(t)\exp\left(-\int_0^t r(t')dt'\right). \qquad (19b)$$

Applying Eq. (19a) to the exponential process (11), we easily find

$$r(t) = A. \qquad (20)$$

In other words, the probability that the first spontaneous emission will occur in the next dt seconds, given that it has not yet occurred by time t, is a constant independent of t. In fact, it is well known that in a stationary Poisson process events occur at a constant rate. Conversely, it is easy to show that if there is a constant probability per unit time that an undecayed atom will decay in the next instant then the atom will have an exponential lifetime distribution.

Applying Eq. (19a) to the long-tailed distribution of Eq. (14a), we find

$$r(t) = a/(1+at). \qquad (21)$$

Equation (21) shows that the rate at which the first photon will arrive in a given coherence volume, given that it has not yet arrived, falls off to zero with increasing time. This decreasing arrival rate results in the corresponding distribution having a long tail. Note, also, that as $at \to \infty$, $r(t) \to 1/t$ and the natural time scale set by $1/a$ is lost. We will return to this point in Sec. VI.

We now present a physical interpretation of Eq. (21). Imagine that the atomic system spontaneously emits photons at a rate A into m coherence volumes (cells) of the electromagnetic field. Because the photons are bosons, the probability that a given photon enters a given coherence volume (cell of phase space) is proportional to 1 plus the number of photons n_i already occupying the cell.[9,10] Hence, the rate at which photons enter a given cell will be

$$r_i(t) = \left[(1+n_i)\ \left(\sum_{j=1}^{m}(1+n_j)\right)^{-1}\right]A, \qquad (22)$$

where the n_j are integer valued random variables whose means are increasing functions of time given by

$$\langle n_j \rangle = At/m = at. \qquad (23)$$

Equation (22) is, of course, conditional on n_i photons having already arrived. Equation (22) can also be expressed in terms of the deviations Δn_j from the mean. Dividing the numerator and denominator of Eq. (22) by m, we have

$$r_i(t) = \left[(1+n_i)\ \left(1+at+\sum_{j=1}^{m}\frac{\Delta n_j}{m}\right)^{-1}\right]a. \qquad (24)$$

Now, the summation term is a random variable with mean zero and variance going to zero at least as fast as $1/m$ (if the n_j were independent, they in fact should be anticorrelated). Hence, for large values of m we have

$$r_i(t) \approx [(1+n_i)/(1+at)]a, \qquad (25)$$

which reduces to Eq. (21) when no photons have arrived yet in the given coherence volume. Note that if we average $r_i(t)$ over the (as yet unspecified) distribution of $n_i(t)$ we obtain, with the help of Eq. (23),

$$\langle r_i(t)\rangle \approx [(1+\langle n_i\rangle)/(1+at)]a = a. \qquad (26)$$

Hence, the average rate of photon arrivals into each cell is a constant, $a = A/m$, as we would expect by conservation of energy.

We thus see that the clumping of photons caused by their boson nature (or equivalently, by the random interference of independently emitted wave packets) makes it less and less likely as time goes on that the next photon emitted by the atomic system will enter an empty coherence volume. Instead, it will be "attracted" (classically, by constructive interference) into those coherence volumes already containing photons. In fact, the rate of arrival of photons into the empty cell diminishes to zero as it falls progressively farther behind its neighboring cells containing many photons. This is the physical origin of the long-tailed distribution for the time until the first photon arrives in a given coherence volume.

More generally, we show in the Appendix that Eq. (25) for arbitrary n_i leads to the Bose–Einstein distribution. However, that is not our main concern here; rather, it is understanding the nonintuitive aspects of the time evolution of the quantum statistics of light.

VI. DISCUSSION

From Eqs. (21) and (25) we see that when the dimensionless product at is not large compared with 1 that $1/a$ (having units of time) sets the natural time scale for the rate function $r(t)$ and hence for the dynamics of our process. However, when $at \to \infty, r(t) \to 1/t$ because the constant a cancels out.[11] We now have the peculiar situation that there is no natural measure for time in the problem (other than t itself). If we were to change our units of time from seconds to years, a plot of $r(t) = 1/t$ would look exactly the same as long as the ordinate and abscissa were appropriately relabeled for the new units (i.e., events/second → events/year and seconds → years). This is an example of what is known as self-similarity, where an object or process exhibits similar features or behavior when viewed at greatly different scales: It is a consequence of the lack of a natural scale for the phenomenon. From Eq. (25) and the results presented in the Appendix we see that the same arguments also apply to the distribution of interarrival times of the later photons as well.[12]

Although long-tailed distributions exhibit counterintuitive properties, we see that they do not really constitute a paradox. In fact, we have seen that their origin can be pictured intuitively in terms of a physically understandable process yielding a decreasing rate function and the loss of any unique time scale for a process. Such distributions do, in fact, arise increasingly often in many physical (and social science) situations, as can be seen in the contemporary literature.[13]

The long-time behavior of deterministic nonlinear dynamical systems[14] in many areas, including optics,[15] fluid dynamics, chemical kinetics, weather,[16] population studies, and physiology has been shown to exhibit not only qualitatively similar behavior in phase space, but also quantitative universal aspects. Consider the region (the attractor) of n-dimensional phase space visited by the system phase point as it evolves in time after any initial transients have died out. For certain ranges of the system parameters, the dimensionality of the subspace containing these points may be integer valued (for example, the two-dimensional surface of a torus in three-space for multiperiodic motion) or fractional valued (for deterministic chaos). In the latter case, the motion is highly complex, being deterministic but not predictable. The set of points containing the path of the phase point, called a strange attractor because of its fractional dimension, is one example of a fractal.[17] One of the most important features of fractals is their self-similarity at all scales. It has been observed that the long-tailed, hyperbolic distributions are those most characteristic of fractals.[18]

It may be somewhat surprising that a type of distribution so much in the forefront of current research interests in nonlinear dynamics also appears in a rather different manner as a result of random interference of light in quantum optics. Such distributions had once been thought to be just pathological curiosities. It now seems worthwhile to understand them better as they become more prevalent in many branches of physics.

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APPENDIX: DERIVATION OF THE BOSE–EINSTEIN DISTRIBUTION FROM $r(t)$

Equation (25) expresses the rate function for the arrival of the $(n+1)$ photon in a given coherence volume or cell of phase space given that n photons have already arrived. We have assumed that $m \gg 1$. We can also write the rate function for the nth photon $(n \to n-1)$ as ($\cap$ means "and")

$$\frac{\Pr[t < T_n \leqslant t + dt \,|\, (T_{n-1} \leqslant t) \cap (T_n > t)]}{dt} = \frac{n}{1+at}a. \qquad \text{(A1)}$$

To obtain the rate for the arrival of the nth photon $\rho_n(t)$ without the condition on T_{n-1} (i.e., the rate function for the total time until the nth photon arrives) we must multiply Eq. (A1) by

$$\Pr(T_{n-1} \leqslant t \,|\, T_n > t) = \Pr[N(t) \geqslant n-1 \,|\, N(t) < n], \qquad \text{(A2)}$$

where we have used the equivalence of events as discussed in Sec. II.

Now, expressing the conditional probability on the right-hand side of Eq. (A2), again using the usual rule, we obtain

$$\begin{aligned}\Pr(T_{n-1} \leqslant t \,|\, T_n > t) &= \Pr\{[N(t) \geqslant n-1] \cap [N(t) < n]\}/\Pr[N(t) < n] \\ &= \Pr[N(t) = n-1]/\Pr[N(t) < n]. \qquad \text{(A3)}\end{aligned}$$

Multiplying Eq. (A3) by Eq. (A1) we have

$$\rho_n(t) = [n/(1+at)]a \times \{\Pr[N(t)=n-1]/\Pr[N(t)<n]\}. \quad (A4)$$

Equation (A4) is based on our physical picture of boson accumulation.

Another expression for $\rho_n(t)$ is obtained by using the general definition of the rate function in terms of the pdf and CDF of T_n:

$$\rho_n(t) = \frac{f_n(t)}{1-F_n(t)} = \left\{-\frac{d}{dt}\Pr[T_n > t]\right\}\Pr[T_n > t]^{-1},$$

$$= \left\{-\frac{d}{dt}\Pr[N(t) < n]\right\}\Pr[N(t) < t]^{-1}. \quad (A5)$$

where we have again used the equivalence of certain events. We can now equate the two expressions for $\rho_n(t)$, cancel terms, and replace $n \to n-1$.

Defining the CDF for $N(t)$ as

$$\hat{F}_t(n) \equiv \Pr[N(t) \leqslant n], \quad (A6)$$

we obtain the pleasingly symmetrical differential–difference equation for the CDF of the number of photons in the coherence volume at time t:

$$(1+n)\nabla\hat{F}_t(n) = -(1+at)\frac{d}{a\,dt}\hat{F}_t(n), \quad (A7)$$

where ∇ is the backward difference operator, i.e., $\nabla g(n) \equiv g(n) - g(n-1)$. Although Eq. (A7) can be solved with standard methods, we already know the solution. It is straightforward to verify that the Bose–Einstein distribution (12) with $\langle N \rangle = at$ does in fact satisfy Eq. (A7).

Two comments are in order at this point. Although our derivation is rather similar to a standard urn model derivation of the Bose–Einstein distribution,[10] ours highlights the stochastic time evolution features of the process in continuous time, whereas the urn model assumes particles are added at discrete, deterministic times. Second, after a simple calculation using Eq. (12) and either Eq. (A4) or (A5), we find that $\rho_n(t) \to 1/t$ as $at \to \infty$ for arbitrary n. Hence, the lack of a natural time scale and self-similarity found for the time until the first photon arrival event also applies more generally for this process.

[1]It was shown by R. Glauber, Phys. Rev. **131**, 2766 (1963) in a semiclassical analysis using the central limit theorem (CLT) that random superposition of a large number of independent wave packets yields a Gaussian distribution for the field and, therefore, a Bose–Einstein distribution of photons. Such a single-mode Gaussian field is often described as pseudothermal or, simply, "thermal." It was shown by E. B. Rockower, Phys. Rev. A **37**, 4319 (1988) and J. Opt. Soc. Am. A **5**, 731 (1988) that in a fully quantum analysis a random superposition of a Poisson number of photons leads to such pseudothermal light, obviating the need to invoke either thermal equilibrium or the CLT.

[2]A. B. Clark and R. L. Disney, *Probability and Random Processes for Engineers and Scientists* (Wiley, New York, 1970) or D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes* (Wiley, New York, 1965). Note, also, that in general, $\langle N(\langle T_1 \rangle)\rangle \neq 1$.

[3]M. Sargent, III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974) or J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[4]E. B. Rockower, N. B. Abraham, and S. R. Smith, Phys. Rev. A **17**, 1100 (1978).

[5]That the linear growth rate cannot realistically continue forever is certainly one counterargument to this. However, the linear growth regime need only last much longer than times t of interest to bring out the nonintuitive aspects of the problem; cf. Sec. IV and Ref. 7.

[6]See the articles by E. B. Rockower in Ref. 1.

[7]It is sufficient to let $c \gg t_0$ in order to bring out the nonintuitive aspects of the situation.

[8]What we call the rate function is often called the hazard rate, failure rate, or event rate function: see, for example, S. Ross, *Introduction to Probability Models* (Academic, New York, 1985), 3rd ed.

[9]F. W. Sears, *Thermodynamics* (Addison-Wesley, Reading, MA, 1953), pp. 317–321.

[10]W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1960), Vol. I, p. 59.

[11]Note that for any hyperbolic distribution for which, as $t \to \infty, \Pr(T > t) = k(1/at)^D$, the corresponding rate function is $r(t) = D/t$. Hence, even when the mean is finite (if $D > 1$) the rate function loses its characteristic time scale and falls off as $1/t$.

[12]One consequence of this [and the divergence of Eq. (13)] is that if we were to plot one realization of the nondecreasing stochastic process $N(t)$ for, say, $N = 0 \to 50$ against the corresponding times where $t \in [0, T_n]$ and then rescale the ordinate and abscissa to $[0,1]$ as described in B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983), pp. 82 and 83 and 286 and 287, we would obtain a type of "devil's staircase" for bosons.

[13]For background to the use of long-tailed distributions in modeling, see B. Mandelbrot, Ref. 12 or B. J. West, *An Essay on the Importance of Being Nonlinear* (Springer-Verlag, Berlin, 1985).

[14]A recent introduction to some of the relevant ideas of dynamical systems theory is presented in H. Kaplan, Am. J. Phys. **55**, 1023 (1987); J. Gleick, *Chaos, Making a New Science* (Viking, New York, 1987); or B. J. West, Ref. 13.

[15]See, e.g., N. B. Abraham, Laser Focus, May 1983 and *Instabilities and Chaos in Quantum Optics*, edited by F. T. Arecchi and R. G. Harrison (Springer-Verlag, New York, 1987).

[16]The classic paper is E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[17]See, for example, D. R. Hofstadter, Sci. Am. **239** (11), 22 (1981); *Chaos*, edited by A. V. Holden (Princeton U.P., Princeton, 1986); J. M. T. Thompson, *Instabilities and Catastrophes in Science and Engineering* (Wiley, New York, 1982); J.-P. Eckmann, Rev. Mod. Phys. **53**, 643 (1981); and E. Ott, Rev. Mod. Phys. **53**, 655 (1981).

[18]See B. Mandelbrot in Ref. 12, p. 341.