OPTICA ACTA, 1979, VOL. 26, NO. 10, 1289–1296

Intensity fluctuations in a two-mode ring laser

EDWARD B. ROCKOWER†

Ketron, Inc., Valley Forge Executive Mall,
Wayne, Pennsylvania 19087, U.S.A.

and N. B. ABRAHAM

Department of Physics, Swarthmore College, Swarthmore,
Pennsylvania 19081, U.S.A.

(*Received 28 November 1978; accepted 13 June 1979*)

Abstract. Intensity fluctuations in a two-mode ring laser with zero detuning are derived using a noise amplification rate-equation model. Both approximate and exact forms of the gain saturation are treated. Steady-state distributions of the intensity are derived analytically permitting calculation of the mean and normalized variance. Numerical solutions yield the time-dependent evolution of these quantities from initial input noise. Recently derived mode competition effects, such as a steady state value of $\frac{1}{3}$ for the normalized variance (rather than zero as in a conventional laser) and negative correlations between the intensity fluctuations of the two modes, appear more simply here and their statistical origin is explained.

1. Introduction

In a recent treatment [1] of a two-mode ring laser a rather unusual result was a normalized variance of $\frac{1}{3}$ for the intensity fluctuations of either mode where there was no detuning of the modes and equal gain for each mode. In this case, the two modes were also shown to have maximum negative cross-correlation.

Considered here is a noise amplification approximation in which the distributed spontaneous emission noise is replaced by thermal input noise [2, 3]. We solve the rate equation for the intensities as a type of stochastic differential equation [4] in which the probabilistic aspects are the initial values of the intensities. The advantage of such a model is a simplification in the analysis. This approximation should generally be applicable when the increase of the intensity of each mode primarily results from gain rather than the distributed noise. The validity of our model is indicated by comparison of our results with those of M-Tehrani and Mandel [1] who used distributed noise in a Fokker–Planck formulation.

Rate equation approaches such as the one taken here neglect coherent interactions between the modes and the atoms [5]. By investigating the behaviour of the intensity rather than the amplitude we also neglect effects caused by the field phase evolution.

Among other results, the fluctuation phenomena mentioned above are also found in our model and are fully explained in terms of the steady-state distribution of the intensity.

†Present address: Lawrence Livermore Laboratory, University of California, P.O. Box 808, Livermore, California 94550, U.S.A.

0030–3909/79/2610 1289 $02·00

2. Ring laser models

The evolution in time of the intensities of two counter-rotating modes in a ring laser is given by†

$$\frac{dx_{1,2}}{dt}=\left[\frac{g_{1,2}x_{1,2}}{1+x_{1,2}+\xi x_{2,1}}\right]-\alpha_{1,2}x_{1,2}, \tag{1}$$

where g is the small signal gain; α, the linear loss coefficient; and ξ, the mode-coupling parameter. The equations are written in terms of intensity normalized to the saturation intensity ($x=I/I_s$).

A common approximation for such a system near threshold is given by

$$\frac{dx_{1,2}}{dt}=[(g_{1,2}-\alpha_{1,2})-g_{1,2}(x_{1,2}+\xi x_{2\ 1})]x_{1,2}. \tag{2}$$

M-Tehrani and Mandel and others make a similar approximation in the amplitude rate equation. Using a renormalized intensity, $y=\frac{1}{2}gx$, and a pump parameter, $a=\frac{1}{2}(g-\alpha)$, yields the form

$$\frac{dy_{1,2}}{dt}=2(a_{1,2}-y_{1,2}-\xi y_{2,1})y_{1,2}. \tag{2 a}$$

Langevin noise sources which represent spontaneous emission noise are also included by M-Tehrani and Mandel [1] on the right hand sides of the amplitude rate equations. In contrast to their work, we represent the early noise by stochastic initial conditions on $y_{1,2}$ (or $x_{1,2}$) and assume that subsequent spontaneous emission is negligible compared with the intensity. While this is not expected to be appropriate in any analysis of the phase of the field, we note that our results for intensity fluctuations should agree with theirs in the limit of large pump parameter (when intensities are much larger than the distributed noise).

We will present analytic results for the steady-state solutions of these equations. We will also explore in detail the temporal evolution of the intensity for the special case when $a_1=a_2$ and $\xi=1$. Numerical calculations of the intensity fluctuations of the modes for this case will be presented and shown to agree with our steady-state results and with those of M-Tehrani and Mandel [1] in the appropriate limits in t and a.

3. General steady-state results

When both modes reach steady state, from either equation (1) or (2 a), we have relations of the form

$$A_1=x_1+\xi x_2 \quad \text{and} \quad A_2=x_2+\xi x_1, \tag{3}$$

where $A=(g-\alpha)/\alpha$ for equation (1) and $A=a$ for equation (2 a). The solution‡ yields

$$x_{1,2}=(A_{1,2}-\xi A_{2,1})/(1-\xi^2), \tag{4}$$

where

$$0\leqslant x_{1,2}\leqslant A_{1,2}.$$

† Similar intensity rate equations are used by Hopf [3] and Casperson [6].

‡ When $\xi=1$ the solution is degenerate requiring the analysis given in the following section.

Thus in general each beam reaches a constant intensity with no fluctuations, so the normalized variance, Q, is zero. These results agree with those of M-Tehrani and Mandel [1]. Retaining the noise sources one can use these results and standard Langevin equation techniques with a quasi-linearization of the amplitude equations to derive the cross correlation between the modes in this limit of large pump parameter.

For the case $A_1 = A_2$ we see that the intensities again go to constant values without fluctuations. However, if $\xi = 1$, $A_1 \neq A_2$, one can see (from equation (2 *a*)) that the stronger mode reaches a steady state value with no fluctuations while the weaker is damped. While our model cannot handle this weak signal it is obvious that the weaker mode contains only the damped spontaneous noise and thus remains thermal. These results also agree with those reported earlier.

4. Symmetrical case: both modes on line centre

From here on we analyse the case $A_1 = A_2 = A$, $\xi = 1$ in detail. Writing using

$$x_1\left(\frac{dx_2}{dt}\right) - x_2\left(\frac{dx_1}{dt}\right)$$

either equation (1) or (2) it is seen that an invariant in time exists for any set of initial conditions: $x_1(t)/x_2(t) = \text{constant}$ in time, and in particular: $x_2(t) = x_1(t)(n_2/n_1)$, where n_1 and n_2 are the initial values of the two mode intensities. Physically, the invariant results from the fact that the instantaneous gain is the same for both modes.

4.1. *Steady-state*

Using the invariant to replace x_2 in equation (3) we obtain

$$x_1 = A(1 + n_2/n_1)^{-1}. \tag{5}$$

From equation (3) the following form of the joint probability density function immediately follows:

$$P(x_1, x_2) = \delta(x_1 + x_2 - A)P(x_1). \tag{6}$$

Using equation (5), the distribution of the intensity fluctuations at equilibrium can be determined from assumptions about the distributions of n_1 and n_2. In particular, if we take both to be negative exponential distributions characteristic of spontaneous emission noise, we find the probability density function

$$P(x_1) = \frac{A}{\bar{n}_1 \bar{n}_2}\left[\frac{A}{\bar{n}_2} - x_1\left(\frac{1}{\bar{n}_2} - \frac{1}{\bar{n}_1}\right)\right]^{-2}. \tag{7}$$

Recall that $0 \leqslant x_1 \leqslant A$. Letting $\gamma = \bar{n}_2/\bar{n}_1$, the ratio of the initial mean values of the two mode intensities, we can calculate the mean intensity, $\bar{x}_1$, and normalized variance, Q_1, of this steady-state distribution:

$$\bar{x}_1 = A[\gamma \ln \gamma + 1 - \gamma](1 - \gamma)^{-2} \tag{8}$$

and

$$Q_1 \equiv \frac{\overline{(\Delta x_1)^2}}{\bar{x}_1^{\,2}} = \left(\frac{(1-\gamma)[1 - \gamma^2 + 2\gamma \ln \gamma]}{[1 - \gamma + \gamma \ln \gamma]^2}\right) - 1. \tag{9}$$

Taking limits on γ we find that

$$\text{if} \quad \gamma \to \infty, \quad \bar{x}_1 \to 0 \quad \text{and} \quad Q_1 \to \infty; \tag{10a}$$

$$\text{if} \quad \gamma = 1, \quad \bar{x}_1 = \tfrac{1}{2}A \quad \text{and} \quad Q_1 = \tfrac{1}{3}; \tag{10b}$$

and

$$\text{if} \quad \gamma \to 0, \quad \bar{x}_1 \to A \quad \text{and} \quad Q_1 \to 0. \tag{10c}$$

From equation (6) we can calculate the cross-correlation function,

$$C \equiv \frac{\overline{x_1 x_2} - \bar{x}_1 \bar{x}_2}{\sqrt{[\overline{(\Delta x_1)^2}\,\overline{(\Delta x_2)^2}]}} = -1, \tag{11}$$

which is true for any value of γ.

When $\gamma = 1$, equation (7) yields

$$P(x_1) = A^{-1}. \tag{12}$$

This is a uniform distribution for the intensity from $x_1 = 0$ to its maximum value $x_1 = A$. For such distributions the normalized variance is $\frac{1}{3}$. Using equation (6) and the results for a uniform distribution, one also shows

$$\frac{\overline{x_1 x_2} - \bar{x}_1 \bar{x}_2}{\bar{x}_1 \bar{x}_2} = -\tfrac{1}{3}. \tag{13}$$

The results in equations (10), (11), and (13) agree with those of M-Tehrani and Mandel. We have shown that the values for $\gamma = 1$ are a consequence of a uniform distribution for the intensity and have extended the derivations beyond the approximate forms of the rate equations to the more exact forms. Thus this form of mode competition should be observable in ring lasers well above threshold. In addition, one now sees that when $a_1 = a_2$ and $\xi = 1$, equation (19) of M-Tehrani and Mandel [1] agrees with our prediction of a uniform distribution for $P(I_1)$, since in the limit of large pump parameters $I_1 + I_2 \approx a$.

Our prediction that Q can become greater than unity indicates that the weaker mode exhibits strong pulsations. The imbalance in the inputs might be experimentally investigated by injecting an external thermal signal into one mode of the ring laser as it is turned on.

4.2. *Temporal evolution*

One can solve equation (2 *a*) explicitly with the result

$$y_1 = \left[\frac{1}{a}\left(1 + \frac{n_2}{n_1}\right) + \frac{1}{n_1} \exp(-2at)\left(1 - \frac{n_1 + n_2}{a}\right)\right]^{-1}. \tag{14}$$

The mean intensity and normalized variance can be calculated as functions of time given the deterministic relationship $y_1(n_1, n_2, t)$. The evolution of the probability density function is given by the formula

$$P(y_1, t) = \int_0^\infty \int_0^\infty \delta(y_1 - y_1(n_1, n_2, t)) P_1(n_1) P_2(n_2)\, dn_1 dn_2, \tag{15}$$

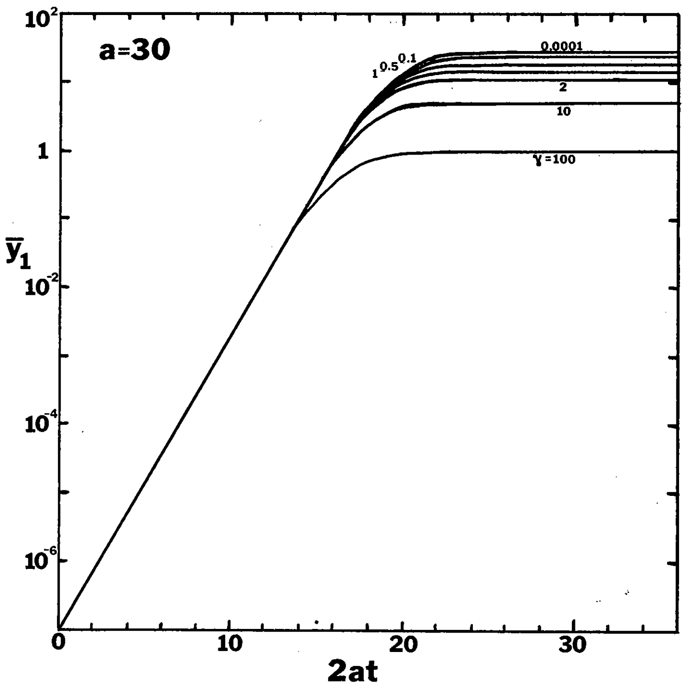

Figure 1. Mean intensity versus $2at$. The initial value $\bar{n}_1$ was held fixed while $\gamma = \bar{n}_2/\bar{n}_1$ was varied as shown. The results were calculated using $N = 30$ (i.e. 900 points) in the Gauss–Laguerre summation.

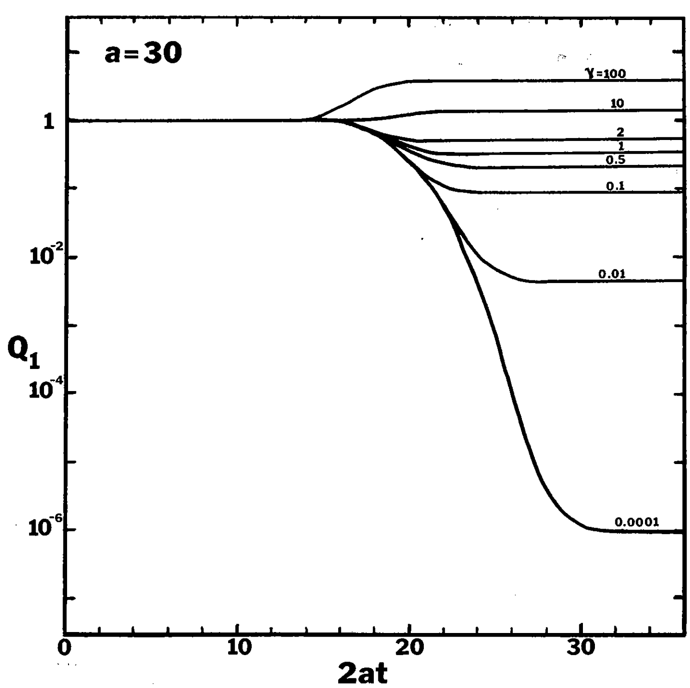

Figure 2. Normalized variance versus $2at$.

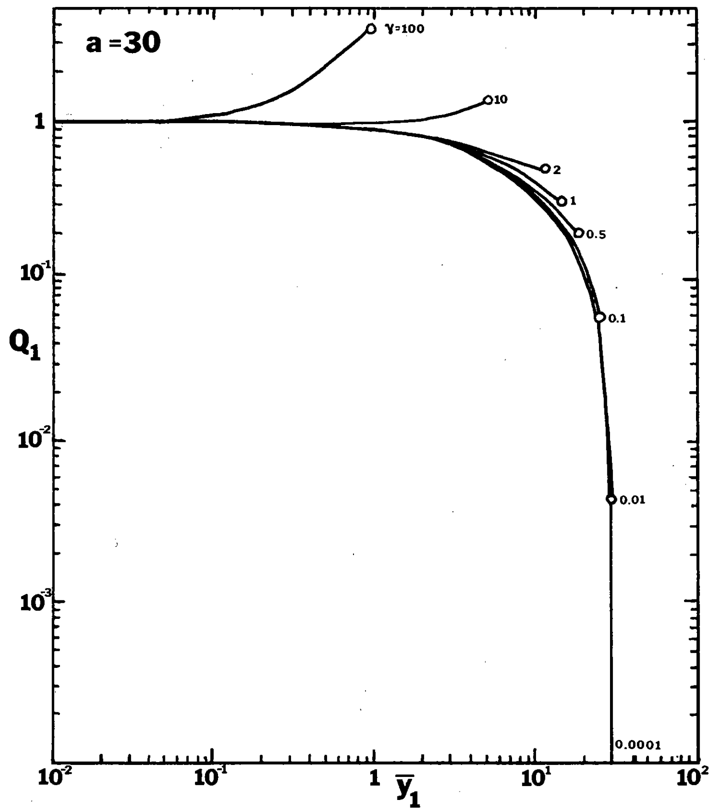

Figure 3. Normalized variance versus mean intensity. The end point indicated on each curve, marked by ○, indicates the asymptotic values.

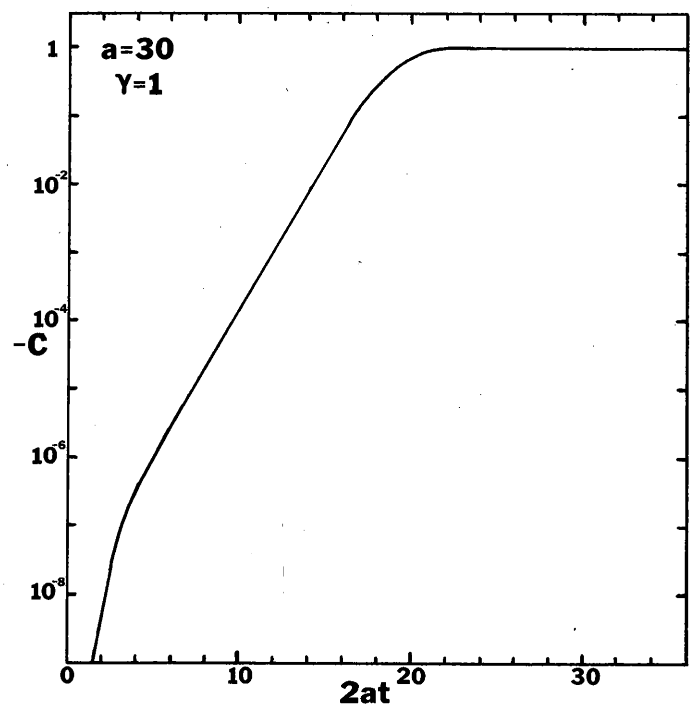

Figure 4. Cross-correlation versus $2at$ for $\gamma=1$.

and the moments by

$$\overline{[y_1(t)]^m} = \int_0^\infty \int_0^\infty [y_1(n_1, n_2, t)]^m P_1(n_1) P_2(n_2)\, dn_1 dn_2. \quad (16)$$

Since we have taken $P_1(n_1)$ and $P_2(n_2)$ to be negative exponential distributions we can integrate equation (16) by Gauss–Laguerre quadratures† to calculate the moments. The results for the mean intensity, normalized variance and cross-correlation are displayed in figures 1–4.

The numerical results are consistent with the analytic results for the steady state. We found that when the intensities were scaled to a, the temporal evolution of $\bar{y}/a$ and Q and the relation of Q to $\bar{y}/a$ became independent of a.

When $\gamma \ll 1$ we see that the normalized variance rapidly approaches its limiting value, $Q \ll 1$. This corresponds to the evolution of the single mode laser. In general, however, mode competition limits this damping of the fluctuations. The output of the symmetrical ring laser described by $\gamma = 1$ clearly does not evolve to a coherent state. The examples for $\gamma > 1$ show that Q_1 reaches values above $\frac{1}{3}$ and indeed may rise above the value of unity characteristic of thermal fluctuations. A report of excess fluctuations in the weaker of two modes [7] might be explained by this calculation and merits further investigation.

5. Discussion

The usefulness of the noise amplification model is demonstrated by the agreement with previous distributed noise formulations and our extension to more general physical conditions. The results have been obtained more simply and the physical interpretation of the results is clearer. The success of our model indicates that it is the variability in the early spontaneous noise (which is carried along to the variability in the steady-state) which is important for the intensity fluctuations rather than the later spontaneous noise (which one knows, is important for the phase diffusion and hence the linewidth). Although our statistical results are achieved by taking ensemble averages over the initial conditions the results agree quite well with those of M-Tehrani and Mandel [1] where it is the Langevin noise terms which cause the fluctuations at any time t.

We see that the special case of the symmetrical ring laser on line centre stands uniquely as one in which mode competition sustains a high level of intensity fluctuations. One may doubt whether this state is physically realizable given normal fluctuations in the physical parameters. It seems an appropriate case for further experimental study.

Because of the nature of our model we have been able to investigate unequal input signals, a variation of parameters not analysed by M-Tehrani and Mandel [1]. Our procedure can easily be adapted to predict results for other inputs such as a coherent

†We use the cartesian direct product formula [8]:

$$\iint F(y, z) \exp(-y) \exp(-z)\, dy dz = \sum_{ij}^{N} F(x_i, x_j) w_i w_j,$$

where the roots (x_i) and weights (w_i) are tabulated by Stroud and Secrest [9]. For our final calculations, that choice of N was made which seemed to give 1 per cent accuracy as determined by extrapolation from results for smaller values of N. This approximate accuracy was confirmed by comparison with the analytic formula for the asymptotic values.

signal from an external single-mode laser. Experimental investigation of injected signals may be worthwhile in light of the predicted pulsations arising from unequal thermal sources.

One should also note that this model is generally applicable to the evolution of two modes in one variable (in this case time). The results may thus apply to the evolution with distance of two modes travelling in one direction in a laser amplifier. Although the non-steady-state results of the present theory are limited by the assumption that the coupling constant between the modes is equal to one, application to the two polarizations in a unidirectional amplifier may be appropriate. In the following paper [10], we present similar theoretical work relating to a bidirectional laser amplifier.

Acknowledgments

We wish to acknowledge useful discussions with S. R. Smith and valuable assistance with the numerical calculations from John Huang. This work was supported in part by a grant from Research Corporation. Some of the materials incorporated in this work were developed with the financial support of the National Science Foundation Grant ENG77-01118.

Mit Hilfe eines Ratengleichungsmodells für Rauschverstärkung werden Intensitätsfluktuationen in einem nicht verstimmten Zweimoden-Ringlaser hergeleitet. Sowohl Näherungen als auch exakte Formen der Verstärkungssättigung werden behandelt. Stabile Intensitätsverteilungen werden analytisch hergeleitet und erlauben die Berechnung des Mittelwerts und der normierten Varianz. Numerische Lösungen geben die zeitabhängige Entwicklung dieser Größen, beginnend beim Eingangsrauschen. Kürzlich hergeleitete Effekte der konkurrierenden Moden-Wechselwirkung, wie ein stabiler Wert von $\frac{1}{3}$ für die normierte Varianz (im Gegensatz zu Null beim konventionellen Laser) und negative Korrelationen der Intensitätsfluktuationen der zwei Moden, ergeben sich hier zwangloser; ihr statistischer Ursprung wird erklärt.

References

[1] M-Tehrani, M., and Mandel, L., 1978, *Phys. Rev.* A, **17,** 677.
[2] Menegozzi, L., and Lamb, W. E., Jr., 1978, *Phys. Rev.* A, **17,** 701.
[3] Hopf, F. A., 1975, Optical Sciences Center Report.
[4] Van Kampen, N. G., 1976, *Physics Lett.* C, **24,** 171.
[5] Risken, H., and Nummedal, K., 1968, *J. appl. Phys.*, **39,** 4662.
[6] Casperson, L. W., 1977, *J. appl. Phys.*, **48,** 256.
[7] M-Tehrani, M., and Mandel, L., 1978, *Phys. Rev.* A, **17,** 694.
[8] Stroud, A. H., 1971, *Approximate Calculation of Multiple Integrals* (Englewood Cliffs: Prentice Hall).
[9] Stroud, A. H., and Secrest, D., 1966, *Gaussian Quadrature Formulas* (Englewood Cliffs: Prentice Hall).
[10] Abraham, N. B., and Rockower, E. B., 1979, *Optica Acta*, **26,** 1297.